GV/DT Invariants and the Weak Gravity Conjecture

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GV/DT invariants play an important role in verifying certain physics conjectures on Quantum Gravity - especially the (Tower) Weak Gravity Conjecture.

This is one example of many profound implications of the Swampland Programme for compact (string) geometry.

We prove⋆ the asymptotic Tower Weak Gravity Conjecture
in 5d N=1 M-theory compactifications.

This is one example of many
important applications of Kähler geometry
and enumerative geometry in physics.

Tower Weak Gravity Conjecture

initiated in [Arkani-Hamed,Motl,Nicolis,Vafa'06]

Consider a gauge theory coupled to quantum gravity, (for simplicity) with abelian gauge factors $U(1)$ and charge lattice $\Lambda_{\mathbf{Q}}$. Then every ray in the lattice $\Lambda_{\mathbf{Q}}$ must support a tower of super-extremal states.

Significance:

- Physics:
  Prediction of existence of certain states with a suitable ratio of charge to mass.
- Mathematics:
  In string/M-theory can be translated into non-vanishing of certain enumerative invariants on compact CYs.

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Consider a gauge theory coupled to quantum gravity, (for simplicity) with abelian gauge factors $U(1)$ and charge lattice $\Lambda_{\mathbf{Q}}$. Then every ray in the lattice $\Lambda_{\mathbf{Q}}$ must support a tower of super-extremal states.

Two notions of **super-extremality**:

1. with respect to extremal black hole:
$$\frac{g_{\mathrm{YM}}^2 q^2}{m^2} \geq \frac{g_{\mathrm{YM}}^2 Q^2}{M^2}\Big|_{\mathrm{B.H.}}$$
2. self-repulsive:
$$F_{\mathrm{Coulomb}} \geq F_{\mathrm{Grav.}} + F_{\mathrm{Yukawa}}$$

F_{Yukawa} in presence of massless scalars - first pointed out in [Palti,'17]

In general both notions are not equivalent [Heidenreich,Reece,Rudelius,'19]
but in asymptotic weak coupling limit they are [Lee,Lerche,TW'18]

Tower Weak Gravity Conjecture

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Consider a gauge theory coupled to quantum gravity, (for simplicity) with abelian gauge factors $U(1)$ *and charge lattice* $\Lambda_{\mathbf{Q}}$. *Then every ray in the lattice* $\Lambda_{\mathbf{Q}}$ *must support a tower of super-extremal states.*

Two notions of **super-extremality**:

1. with respect to extremal black hole:

$$\frac{g_{\rm YM}^2 q^2}{m^2} \geq \frac{g_{\rm YM}^2 Q^2}{M^2}\Big|_{\rm B.H.}$$

2. self-repulsive:

$$F_{\rm Coulomb} \geq F_{\rm Grav.} + F_{\rm Yukawa}$$

Tower: [Heidenreich,Reece,Rudelius'15-16] [Montero,Shiu,Soler'16] [Andriolo et al.'16]

$\exists$ super-extremal particle of charge $n\mathbf{Q}$ for $n \in \mathcal{I}$ an *infinite* set

WGC: BPS Towers

Fact: Every BPS state automatically (super-)extremal

$\Longrightarrow$ Tower of BPS states sufficient for tower WGC

[Grimm,Palti,Valenzuela'18] [Gendler,Valenzuela'20] [Grimm,Heisteeg'20] ...

[Alim,Heidenreich,Rudelius'21] [Gendler,Heidenreich,McAllister,Moritz,Rudelius'22]

What if there is no BPS tower in a certain direction in charge lattice?

This is the generic situation.

Examples:

F-theory compactification

- to 6d $N = (1, 0)$ [Lee,Lerche,TW'18]
- to 4d $N = 1$ [Lee,Lerche,TW'19] [Kläwer,Lee,TW,Wiesner'20] [Heidenreich,Reece,Rudelius'21] [Cota,Mininno,TW,Wiesner'22 (1)]

M-theory compactification to 5d $N = 1$ [Alim,Heidenreich,Rudelius'21] [Cota,Mininno,TW,Wiesner'22 (2)]

Beyond BPS Towers

Main result: [Cota,Mininno,TW,Wiesner'22 (2)]

Consider M-theory compactified on CY_3 to 5d $N = 1$.
Whenever a direction in the charge lattice does not admit a (super-extremal) BPS tower, then

- either there is no weak coupling limit for the relevant U(1)s
- or there does exist a super-extremal non-BPS tower.

$\Longrightarrow$ Establishes *asymptotic* tower WGC in 5d M-theory.

weak coupling limit		Kähler geometry of CY3 and its infinite distance limits
	$\Longleftrightarrow$	
state counting		GV/DT invariants

Reminder: M-theory

11D supergravity

$$S_{11d} = \frac{M_{11}^9}{2} \int_{\mathbb{R}^{1,10}} R \star 1 - \frac{M_{11}^9}{2} \int_{\mathbb{R}^{1,10}} (dC_3) \wedge \star dC_3 + \ldots$$

C_3 : 3-form gauge field

Two types of branes:

- Membrane M2 (2+1 dimensional) couples to C_3 electrically

$$S_{\mathrm{M2}} = \int_{\mathrm{M2}} \iota^* C_3 + \ldots$$

- 5-brane M5 (5+1 dimensional) magnetic dual of M2

$$S_{\mathrm{M5}} = \int_{\mathrm{M5}} \iota^* C_6 + \ldots \qquad dC_3 = * dC_6$$

Compactify on CY3 X_3: $\mathbb{R}^{1,10} = \mathbb{R}^{1,4} \times X_3$

WGC in 5d M-theory

$\mathbb{R}^{1,10} = \mathbb{R}^{1,4} \times X_3$: Effective 5d N=1 theory (8 supercharges)

- Basis of U(1) gauge groups from expanding

$$C_3 = A^\alpha \wedge J_\alpha \,, \qquad \alpha = 1, \ldots, h^{1,1}(X_3)$$

- J_α: Basis of Kähler cone generators $J = v^\alpha J_\alpha$
- Gauge kinetic terms

$$S_{5d} = \frac{M_{\rm Pl}^3}{2} \int_{\mathbb{R}^{1,4}} R \star 1 - \frac{1}{2g_5^2} \int_{\mathbb{R}^{1,4}} f_{\alpha\beta} F^\alpha \wedge \star F^\beta + \ldots \qquad F^\alpha = dA^\alpha$$

Gauge kinetic matrix $f_{\alpha\beta} \Longleftrightarrow$ Kähler moduli (modulo overall volume)

$$f_{\alpha\beta} = \frac{1}{\mathcal{V}^{1/3}} \int_{X_3} J_\alpha \wedge \star J_\beta = \left(\hat{\mathcal{V}}_\alpha \hat{\mathcal{V}}_\beta - \hat{\mathcal{V}}_{\alpha\beta} \right)$$

$$\mathcal{V} = \frac{1}{6} \int_{X_3} J^3 \,, \qquad \mathcal{V}_\alpha = \frac{1}{2\mathcal{V}} \int_{X_3} J_\alpha \wedge J^2 \,, \qquad \mathcal{V}_{\alpha\beta} = \frac{1}{\mathcal{V}} \int_{X_3} J_\alpha \wedge J_\beta \wedge J \,, \qquad \hat{v}^\alpha = \frac{v^\alpha}{\mathcal{V}}$$

WGC in 5d M-theory

Self-repulsiveness condition for states of

- charges Q_α under $U(1)_\alpha$
- Kähler moduli dependent mass $M_k(v^\alpha)$

$$F_{\text{Coulomb}} \overset{!}{\geq} F_{\text{grav}} + F_{\text{Yukawa}}$$

$$\frac{(M_{\text{Pl}} g_5^2)(Q_\alpha f^{\alpha\beta} Q_\beta)}{M_k^2/M_{\text{Pl}}^2} \overset{!}{\geq} \left.\frac{d-3}{d-2}\right|_{d=5} + \frac{1}{2}\frac{M_{\text{Pl}}^4}{M_k^4}\left(f^{\alpha\beta} - \frac{1}{3}\hat{v}^\alpha \hat{v}^\beta\right)\partial_\alpha\left(\frac{M_k^2}{M_{\text{Pl}}^2}\right)\partial_\beta\left(\frac{M_k^2}{M_{\text{Pl}}^2}\right)$$

basis of U(1) gauge groups from expanding

$$C_3 = A^\alpha \wedge J_\alpha\,, \qquad \alpha = 1, \ldots, h^{1,1}(X_3)$$

J_α: Basis of Kähler cone generators $J = v^\alpha J_\alpha$

$$S_{5d} = \frac{M_{\text{Pl}}^3}{2}\int_{\mathbb{R}^{1,4}} R \star 1 - \frac{1}{2g_5^2}\int_{\mathbb{R}^{1,4}} f_{\alpha\beta} F^\alpha \wedge \star F^\beta + \ldots$$

WGC: BPS Towers

First source for (super-)extremal tower:

BPS particles in 5d from M2-branes on holomorphic curves on X_3

Reminder:

M2 $= \mathbb{R} \times C \subset \mathbb{R}^{1,5} \times X_3$

$$\Longrightarrow S_{\mathrm{M2}} = \int_{\mathbb{R}\times C} \iota^* C_3 = \int_{\mathbb{R}} A^\alpha \int_C J_\alpha$$

$$= Q_\alpha \int_{\mathbb{R}} A^\alpha \qquad Q_\alpha : \text{charge}$$

Mass of particle: $m = \mathrm{vol}(C)$

holomorphic $C \Longleftrightarrow$ charged BPS particle $\Longrightarrow$ (super-)extremality

WGC: BPS Towers

First source for (super-)extremal tower:

BPS particles in 5d from M2-branes on holomorphic curves on X_3

Conjecture: [Alim,Heidenreich,Rudelius'21]

Every curve class $C \in \mathrm{Mov}_1(X_3)$ supports a tower of BPS states, i.e.

$$N_{(g=0)}(nC) \neq 0 \qquad \forall C \in \mathrm{Mov}_1(X_3)$$

Recall: Movable curve cone $\mathrm{Mov}_1(X_3)$ is dual to cone of effective divisors $\mathsf{Eff}^1(X_3)$

✓ confirmed in many examples in [Alim,Heidenreich,Rudelius'21][Gendler et al.'22]

✓ Black hole extremality condition $=$ BPS condition
for such $C \in \mathrm{Mov}_1(X_3)$ [Alim,Heidenreich,Rudelius'21]

Challenge for tower WGC: What if there are no BPS towers?

Example: Conifold

Beyond BPS Towers

Main result: [Cota,Mininno,TW,Wiesner'22]

Whenever there is no BPS tower, then

- either there is no weak coupling limit for the U(1)s
- or there does exist a super-extremal non-BPS tower.

Strategy:

1. Characterise all weak coupling limits $\Longleftrightarrow$ Kähler geometry
2. Identify towers of super-extremal BPS or non-BPS states for U(1)s with a weak coupling limit
   $\Longleftrightarrow$ DT invariants/Noether-Lefschetz theory

Weak Coupling Limits

$$S_{5d} = \frac{M_{\text{Pl}}^3}{2}\int_{\mathbb{R}^{1,4}} R \star 1 - \frac{1}{2g_5^2}\int_{\mathbb{R}^{1,4}} f_{\alpha\beta} F^\alpha \wedge \star F^\beta + \ldots$$

Gauge kinetic matrix $f_{\alpha\beta} \Longleftrightarrow$ Kähler moduli (at fixed overall volume)

$$f_{\alpha\beta} = \frac{1}{\mathcal{V}^{1/3}}\int_{X_3} J_\alpha \wedge \star J_\beta = \left(\hat{\mathcal{V}}_\alpha \hat{\mathcal{V}}_\beta - \hat{\mathcal{V}}_{\alpha\beta}\right)$$

1) Necessary condition for weak coupling limit:

Entries of $f_{\alpha\beta} \to \infty \leftrightarrow$ Infinite distance limits (in Kähler moduli space at fixed overall volume $\mathcal{V}$) cf. [Heidenreich,Rudelius'20]

2) More precise criterion:

$U(1)_C = c_\alpha U(1)^\alpha$ basis $U(1)^\alpha \leftrightarrow A^\alpha$ in $C_3 = A^\alpha J_\alpha$

$$\Lambda^2_{\text{WGC}}\left(U(1)_C\right) = g^2_{\text{YM,C}} M^3_{\text{Pl}} = g_5^2\left(c_\alpha f^{\alpha\beta} c_\beta\right) M^3_{\text{Pl}}$$

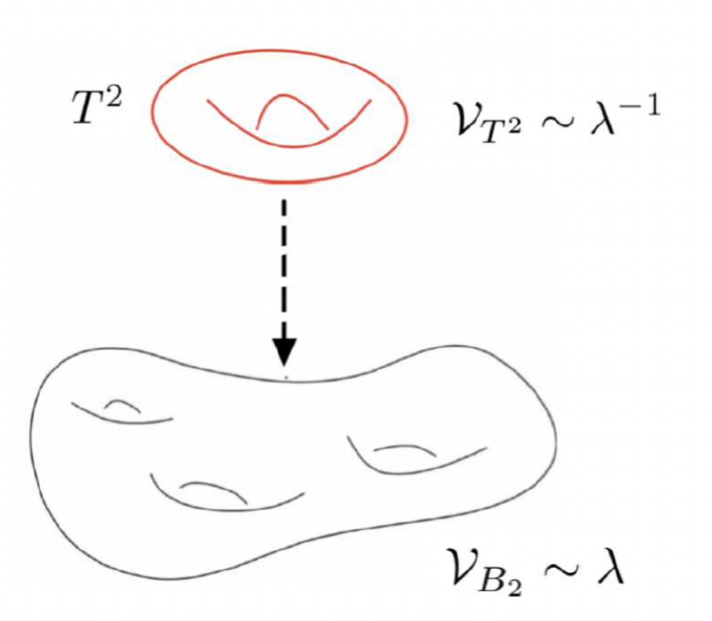
$$\frac{\Lambda^2_{\text{WGC}}\left(U(1)_C\right)}{\Lambda^2_{\text{QG}}} \to 0$$

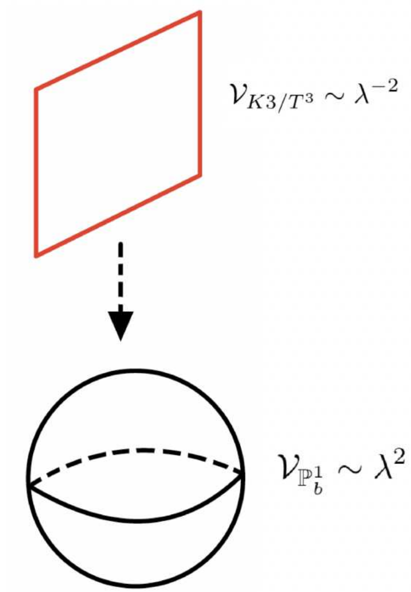
$\Lambda_{\text{QG}} = \Lambda_{\text{sp.}}$: species scale for limit

Weak Coupling Limits

Characterisation of weak coupling limits: [Cota,Mininno,TW,Wiesner'22]

follows from classification of infinite distance limits in classical Kähler moduli space of X_3 at fixed volume [Lee,Lerche,TW'19]:





KK tower from M2 on T^2
Effective decomp. 5d $\rightarrow$ 6d
$\Lambda_{\text{sp.}} = M_{\text{Pl., 6d}}$

heterotic/Type II string from M5 on fiber
5d emergent heterotic/Type II string limit
$\Lambda_{\text{sp.}} = \Lambda_{\text{sp.,string}}$

Weak Coupling Limits

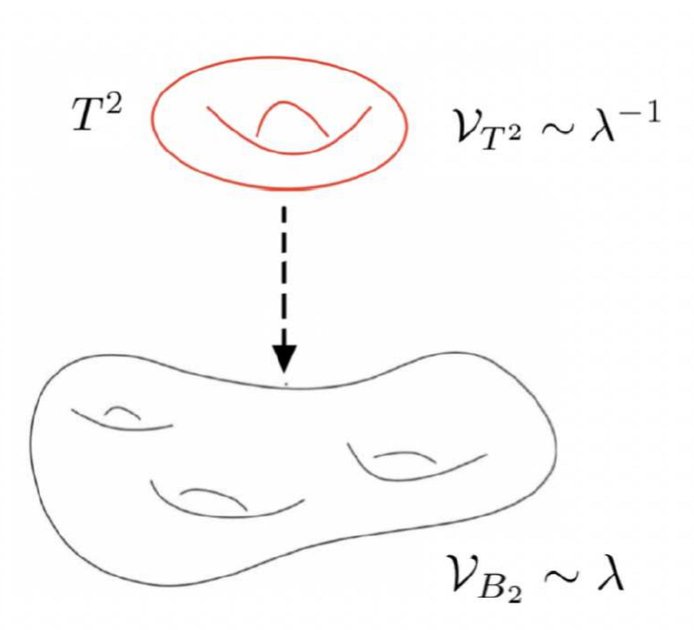
Characterisation of weak coupling limits: [Cota,Mininno,TW,Wiesner'22]

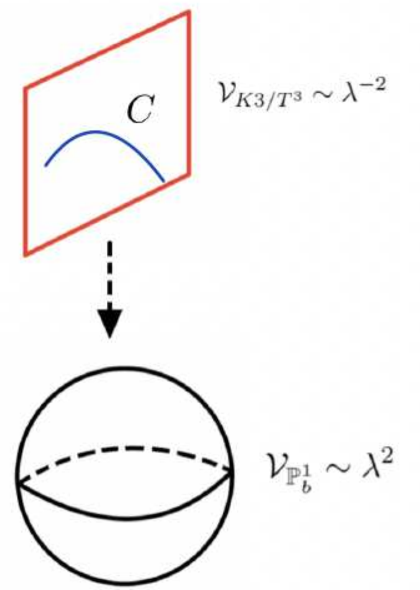
The only $U(1)$s with a weak coupling limit:

$$U(1)_C = c_\alpha U(1)^\alpha \qquad \text{with} \quad C = c_\alpha \omega^\alpha \in H_2(X_3) \quad \text{for}$$

1. $C = T^2$ - a generic torus fiber of X_3
2. $C \subset S$ for S a generic K3 or T^4 fiber of X_3

   or a degenerate such fiber at finite distance in the fiber c.s. moduli space.





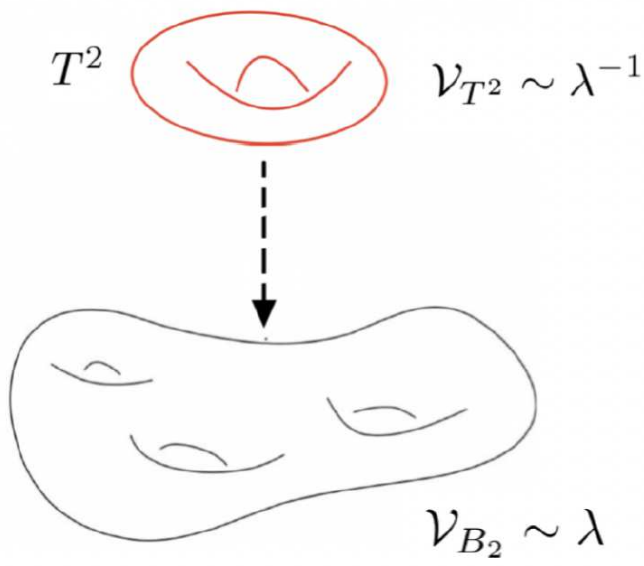
Elliptic tower counting

Suppose X_3 admits fibration $\pi : T^2 \to B_2$ with generic fiber $T^2 \equiv \mathcal{E}$.
Unless X_3 also admits a K3 or T^4 surface:

only U(1) with a weak coupling limit: $U(1)_{\mathcal{E}} = c_\alpha U(1)^\alpha$ $\mathcal{E} = c_\alpha \omega^\alpha$

- Intuition:
  '$g^2_{\text{YM, C}} M_{\text{Pl}} \sim \mathcal{V}_C^{-2}$' $\mathcal{V}_C$: 'dual' divisor to C
- $\mathcal{V}_{T^2} \sim \frac{1}{\lambda}\,, \quad \mathcal{V}_{B_2} \sim \lambda \Longrightarrow g^2_{\text{YM},\mathcal{E}} M_{\text{Pl}} \sim \frac{1}{\lambda^2}$
- $\frac{\Lambda^3_{\text{sp,KK}}}{M^3_{\text{Pl}}} = \left(\frac{M^3_{\text{KK}}}{M^3_{\text{Pl}}}\right)^{\frac{n}{3+n}} \Big|_{n=1}$

  cf.[Montero,Vafa,Valenzuela'22]
- $\frac{M^3_{\text{KK}}}{M^3_{\text{Pl}}} \sim \mathcal{V}^3_{T^2} \sim \frac{1}{\lambda^3} \Longrightarrow \frac{\Lambda^3_{\text{sp,KK}}}{M^3_{\text{Pl}}} \sim \frac{1}{\lambda^{3/4}}$



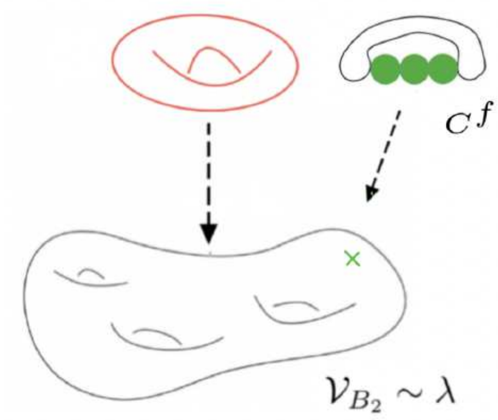
$$\frac{\Lambda^2_{\text{WGC}}(U(1)_{\mathcal{E}})}{\Lambda^2_{\text{QG}}} \sim \frac{g^2_{\text{YM},\mathcal{E}} M^3_{\text{Pl}}}{\Lambda^2_{\text{sp,KK}}} \sim \frac{1/\lambda^2}{1/\lambda^{1/2}} \sim \frac{1}{\lambda^{3/2}} \to 0 \quad \checkmark$$

Elliptic tower counting

Unless X_3 also admits a K3 or T^4 surface:

- *only* $U(1)_\mathcal{E}$ has a weak coupling limit
- All other fibral curves not weakly coupled:

$$\frac{g^2_{\mathsf{YM},C^f} M^3_{\mathsf{Pl}}}{\Lambda^2_{\mathsf{sp,KK}}} \sim \frac{1/\lambda^{1/2}}{1/\lambda^{1/2}} \sim 1$$



Super-extremal towers:

- $\exists$ tower of BPS states charged under $U(1)_\mathcal{E}$:
  M2-branes wrapped n-times on T^2 fiber

$$N^0_{n\mathcal{E}} = -\chi(X_3)$$

$\Longrightarrow$ super-extremal BPS tower $\Longleftrightarrow$ asymptotic tower WGC
Interpretation: KK tower for decompactification 5d to 6d

- No BPS towers along other fibral curves, and no non-BPS tower known

Elliptic tower counting

Super-extremal towers:

- $\exists$ tower of BPS states charged under $U(1)_{\mathcal{E}}$:
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  Interpretation: KK tower for decompactification 5d to 6d

- No BPS towers along other fibral curves, and no non-BPS tower known
  $\leftrightarrow$ corresponding gauge group does not become weakly coupled in asymptotic 6d theory (by assumption B_2 not rationally fibered)

  cf. [Cota,Mininno,TW,Wiesner'22 (1)]

- Gauge groups 'from base' become 2-forms in asymptotic 6d theory

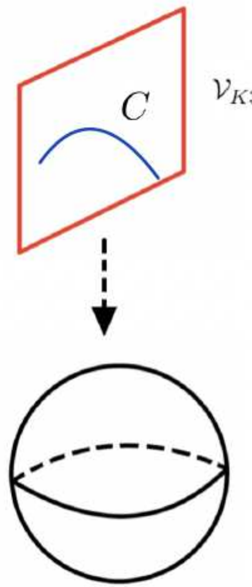
K3 tower counting

Suppose X_3 admits fibration $\rho : K3 \to \mathbb{P}^1$

The $only^*$ U(1) which can undergo a weak coupling limit is

$$U(1)_C = c_\alpha U(1)^\alpha \qquad C = c_\alpha \omega^\alpha$$

for C a curve in a generic K3 fiber or in a special K3 fiber at finite distance in moduli space.



- $g^2_{\mathsf{YM,C}} M_{\mathsf{Pl}} \sim \frac{1}{\lambda^2}$
- Species scale set by tension of emergent heterotic string cf. [Dvali,Lüst'09] [Marchesano,Melotti'22] [Castellano,Herraez,Ibanez'22]

$$\Lambda^2_{\mathsf{sp}} \sim M^2_{\mathsf{het}} \log\left(\frac{M_{\mathsf{Pl}}}{M_{\mathsf{het}}}\right), \qquad M^2_{\mathsf{het}} \sim \frac{1}{\lambda^2} M^2_{\mathsf{11d}}$$

$$\frac{\Lambda^2_{\mathsf{WGC}}(U(1))}{\Lambda^2_{\mathsf{sp}}} \sim \frac{1/\lambda^2}{1/\lambda^2(\log(\lambda))} \to 0$$

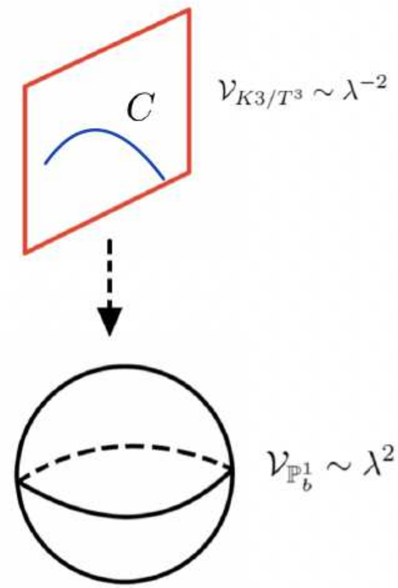
K3 tower counting

Weakly coupled: $U(1)_C$, $C \subset$ generic/finite distance deg. K3

Such curves lie in a lattice

$$\Lambda^*_{\mathbb{R}} = \Lambda^*_+ \oplus \Lambda^*_-$$

$\Lambda^*_{\mathbb{R}}$ lattice of charges with respect to such U(1) of signature $(1, r)$, $r \leq 19$



- $\mathbf{C}^2 \geq 0$: BPS tower exists

  ✓such curves are movable inside K3 fiber and hence in movable cone

  ✓in agreement with BPS index counting via modular forms

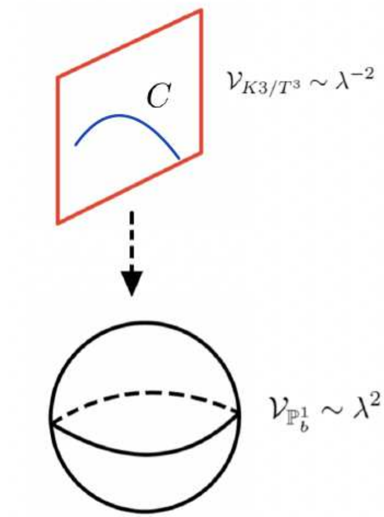
  Gopakumar-Vafa invariants for M2-brane on $C \subset K3$: [Harvey,Moore'99], ...

$$N_C^{g=0} = c\left(\frac{C^2}{2}\right) \qquad f(q) = \sum_{n=-1}^{\infty} c(n) q^n \quad \textbf{modular form}$$

Infinite tower on $nC \leftrightarrow C \cdot C \geq 0 =$ non-contractible curves inside K3

K3 tower counting

Weakly coupled: $U(1)_C$, $C \subset$ generic/finite distance deg. K3



- $\mathbf{C}^2 \geq 0$: BPS tower exists
- $\mathbf{C}^2 < 0$: No BPS tower exists
  ✓curves are rigid inside K3 fiber and hence not in movable cone

Claim: Tower of non-BPS states takes over [Cota,Mininno,TW,Wiesner'22 (2)]

- Special set of states from excitations of MSW-type heterotic string obtained by wrapping M5-brane on K3 fiber is super-extremal.
- Existence established via relation of elliptic genus and 4d BPS invariants in Type IIA on CY3[a]

  [Bouchard,Creutzig,Diaconescu,Doran,Quigley,Sheshmani16] [Pandharipande,Thomas'16]

[a]Analysis most explicitly in absence of multi-components fibers, but expect results to carry over more generally.

K3 tower counting

M-theory on $X_3 \times S^1_M$
bound state of M5-brane on K3
(winding,KK) number (r, n)
M2 brane on curve $\mathbf{Q}$ on K3

Type IIA string on X_3
D4-D2-D0 bound state
of charge vector
$\gamma = (r\Sigma_{\mathrm{K3}}, \mathbf{Q}, n)$

Can view these states as winding modes of heterotic string from M5 on K3 at KK level n and charge vector $\mathbf{Q}$

Special case $r = 1$:
By level matching can identify $n \leftrightarrow n_L =$ (left-moving) excitation level of single heterotic string

4d BPS states of charge vector $\gamma = (\Sigma_{\mathrm{K3}}, \mathbf{Q}, n)$	$\Longrightarrow$	existence of non-BPS string excitations in 5d at level $n_L = n$ and charge $\mathbf{Q}$

Goal: Show existence of state at $n = -\frac{1}{2}\mathbf{Q}^2 \Longleftarrow$ super-extremal (later)

K3 tower counting

Modified elliptic genus of wrapped heterotic string [Gaiotto,Strominger,Yin'06]:

$$Z_{\mathbf{S}}^{(r=1)}(\tau,\bar\tau,\mathbf{z}) = \mathrm{Tr}_{\mathrm{RR}} F_{\mathrm{R}}^2 (-1)^{F_{\mathrm{R}}} q^{L_0-\frac{c_{\mathrm{L}}}{24}} \bar q^{\bar L_0 - \frac{c_{\mathrm{R}}}{24}} e^{2\pi i z^i Q_i}$$
$$= \sum_{n^{(L)},n^{(R)}} N(n^{(L)},n^{(R)},\mathbf{Q})\, q^{n^{(L)}-1}\, \bar q^{n^{(R)}} e^{2\pi i z^i Q_i}$$

Expression in terms of 4d BPS numbers:

$$Z_{\mathbf{S}}^{(r=1)}(\tau,\bar\tau,\mathbf{z}) = \sum_{\boldsymbol{\mu}\in\Lambda^*/\Lambda} Z_{\boldsymbol{\mu}}(\tau)\Theta^*_{\boldsymbol{\mu}}(\tau,\bar\tau,\mathbf{z})$$

$$Z_{\boldsymbol{\mu}}(\tau) = \sum_{n=0}^{\infty} \Omega(\gamma) q^{n+\mathbf{Q}^2/2-1}, \quad \Theta^*_{\boldsymbol{\mu}}(\tau,\bar\tau,\mathbf{z}) = \sum_{\boldsymbol{\lambda}\in\boldsymbol{\mu}+\Lambda} q^{-\frac12(\boldsymbol{\lambda})_-^2}\, \bar q^{+\frac12(\boldsymbol{\lambda})_+^2}\, e^{2\pi i(\boldsymbol{\lambda})\cdot z}$$

$\Omega(\gamma)$: 4d BPS index for D4-D2-D0 states (Donaldson-Thomas invariants)

For simplicity here focus on $\mathbf{Q}=\mathbf{Q}_- \in \Lambda^*_-$

If $\Omega(\gamma)\neq 0$ for $n=-\frac{\mathbf{Q}^2}{2}>0$, then have states at excitaton level n in 5d

Recall:

n: KK number on S^1_M, but by level matching identified with excitation level n_L in 5d

K3 tower counting

Show: $\Omega(\gamma) \neq 0$ for $\gamma = (\Sigma_{\rm K3}, \mathbf{Q}, n)$ at $n = -\dfrac{\mathbf{Q}^2}{2}$, $\mathbf{Q} \in \Lambda_-^*$

Key insight: [Bouchard,Creutzig,Diaconescu,Doran,Quigley,Sheshmani16]

$$\begin{aligned} Z_{\boldsymbol{\mu}}(\tau) &= \sum_{n=0}^{\infty} \Omega(\gamma) q^{n+\mathbf{Q}^2/2-1} \\ &= \eta^{-24}(\tau)\Phi_{\boldsymbol{\mu}}(\tau) = \left[q^{-1} + 24 + \mathcal{O}(q)\right] \Phi_{\boldsymbol{\mu}}(\tau) \end{aligned}$$

for $\Phi_{\boldsymbol{\mu}}(\tau)$ a component of a vector-valued modular form

Expansion coefficients related to Noether-Lefschetz numbers [Maulik,Pandharipande'13] [Pandharipande,Thomas'16]

$$NL_{(h,\mathbf{Q})} = \text{Coeff}\left(\Phi_{\boldsymbol{\mu}}, q^{\Delta_{\rm NL}}\right) , \qquad \Delta_{\rm NL} = \frac{1}{2}\eta^{ij} Q_i Q_j + 1 - h$$

- If $\Delta_{\rm NL} < 0$, then $NL_{(h,\mathbf{Q})} = 0$
- If $\Delta_{\rm NL} = 0$, then $NL_{(h,\mathbf{Q})} = -2$
- If $\Delta_{\rm NL} > 0$, then $NL_{(h,\mathbf{Q})} \in \mathbb{Z}$

$\Longrightarrow$ States with $n = -\frac{\mathbf{Q}^2}{2}$ appear at order q^{-1} in

$$Z_{\mathbf{0}}(\tau) = \eta^{-24}(\tau)\Phi_{\mathbf{0}}(\tau) = \left[q^{-1} + 24 + \mathcal{O}(q)\right] \left[-2 + \mathcal{O}(q)\right] = -2q^{-1} + \mathcal{O}(q^0)$$

Super-extremality

Remains to show:

Non-BPS states at $n_k = -\frac{1}{2}\mathbf{Q}^2$ are self-repulsive/super-extremal in the asymptotic weak coupling limit.

$$
\begin{array}{rcl}
F_{\text{Coulomb}} & \overset{!}{\geq} & F_{\text{grav}} \qquad + \qquad F_{\text{Yukawa}} \\
\dfrac{(M_{\mathsf{Pl}} g_5^2)(Q_\alpha f^{\alpha\beta} Q_\beta)}{M_k^2/M_{\mathsf{Pl}}^2} & \overset{!}{\geq} & \left.\dfrac{d-3}{d-2}\right|_{d=5} + \dfrac{1}{2}\dfrac{M_{\mathsf{Pl}}^4}{M_k^4}\left(f^{\alpha\beta} - \dfrac{1}{3}\hat{v}^\alpha \hat{v}^\beta\right)\partial_\alpha\left(\dfrac{M_k^2}{M_{\mathsf{Pl}}^2}\right)\partial_\beta\left(\dfrac{M_k^2}{M_{\mathsf{Pl}}^2}\right)
\end{array}
$$

Super-extremality

$$
\begin{array}{ccccc}
F_{\text{Coulomb}} & \overset{!}{\geq} & F_{\text{grav}} & + & F_{\text{Yukawa}} \\
\frac{(M_{\mathsf{Pl}} g_5^2)(Q_\alpha f^{\alpha\beta} Q_\beta)}{M_k^2 / M_{\mathsf{Pl}}^2} & \overset{!}{\geq} & \left.\frac{d-3}{d-2}\right|_{d=5} & + & \frac{1}{2}\frac{M_{\mathsf{Pl}}^4}{M_k^4}\left(f^{\alpha\beta} - \frac{1}{3}\hat{v}^\alpha \hat{v}^\beta\right)\partial_\alpha\left(\frac{M_k^2}{M_{\mathsf{Pl}}^2}\right)\partial_\beta\left(\frac{M_k^2}{M_{\mathsf{Pl}}^2}\right)
\end{array}
$$

Explicitly check this for states at excitation level $n_k = -\frac{1}{2}\mathbf{Q}^2$

Input from string theory:

$$M_k^2 = 8\pi(n_k - 1)T_s + \Delta_{\mathsf{CB}}$$

- First term: Contribution from string oscillators, with string tension

$$T_s = 2\pi \mathcal{V}_{\mathbf{S}} M_{\mathrm{11d}}^2 = 2\pi (4\pi)^{-2/3} \hat{\mathcal{V}}_{\mathbf{S}} M_{\mathsf{Pl}}^2$$

- Δ_{CB}: contribution from Coulomb branch in 5d

$$\Delta_{\mathsf{CB}} = 4\pi^2 (4\pi)^{-2/3} Q_i Q_j \hat{v}^i \hat{v}^j \, M_{\mathsf{Pl}}^2 \qquad \hat{v}^i \text{ : Kähler moduli of K3 fiber}$$

In the asymptotic limit a number of simplifications occur.

$\Longrightarrow$ **Together with $n_k = -\frac{1}{2}\mathbf{Q}^2$ the inequality is marginally obeyed**

Conclusions

Asymptotic Tower Weak Gravity Conjecture in 5d M-theory

Non-BPS tower from emergent string excitations yield WGC tower in potentially weakly coupled directions of charge lattice not associated with a KK tower

✓ Similar conclusions also in F-theory with 6d N=1 or 4d N=1

Technical question:

Improve understanding of T^4 fibrations and asymptotic Type II theories

Conceptual questions:

What if there is no (known) tower in a certain direction?

We have shown that this means: $\frac{\Lambda_{\mathsf{WGC}}}{\Lambda_{\mathsf{QG}}} \geq 1$

$\Longrightarrow$ Does it even make sense to define a tower of states in the EFT in such theories?