Mathematical Aspects of (Bosonic) String Theory

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Contents

Note to the reader:

These lecture notes are a typed up version of Dr. Simone Noja's handwritten notes from the course "Mathematical Aspects of String Theory" he teached during the winter semester 2022/2023 at the university of Heidelberg. The courses were meant as a mathematical supplement to the lectures in String Theory held by Prof. Johannes Walcher during the same time. Please note that there are probably a lot of typos everywhere!

A main reference is the textbook "Complex Geometry: An Introduction" by Daniel Huybrechts.

1 Elements of Complex Analysis

1.1 Elementary Characterisations

Definition 1 (Analytic Function)**.** Let $U \subseteq \mathbb{C}$ be an open subset (in the complex topology). We say that $F : U \to \mathbb{C}$ is *analytic* in U if $\forall z_0 \in U\ \exists B_\varepsilon(z_0)$ such that F has a Taylor series expansion in $z - z_0$, i.e.

$$F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{with} \quad a_n \in \mathbb{C}\ \forall n \in \mathbb{N}_0 \tag{1}$$

converges uniformly and absolutely.

Remark. Representing $\mathbb{C}^n \cong \mathbb{R}^n \oplus i\mathbb{R}^n$ one writes $\underline{z} = \underline{x} + i\underline{y}$ with $(\underline{x}, \underline{y}) \in \mathbb{R}^n \oplus \mathbb{R}^n$. In particular $\mathbb{C} \ni z = x + iy$.

It follows that $F : U \to \mathbb{C}$ can be considered as complex functions of two real variables:

$$F(z) = F_{\mathbb{C}}(x, y) = u(x, y) + iv(x, y) \tag{2}$$

with $u : U_{\mathbb{R}} \to \mathbb{R}$ and $v : U_{\mathbb{R}} \to \mathbb{R}$.

Definition 2 (Holomorphic Function)**.** Let $U \subseteq \mathbb{C}$ be an open set. We say that $F : U \to \mathbb{C}$ with $F(z) = u(x, y) + iv(x, y)$ is a *holomorphic function* if there are $u, v \in \mathcal{C}^0_{\mathbb{R}}$ such that they satisfy the following system of PDE's:

$$\partial_x u = \partial_y v \qquad \text{and} \qquad \partial_y u = -\partial_x v \tag{3}$$

These are the *Cauchy-Riemann-Equations.*

Note. Requiring $u, v \in \mathcal{C}^\infty$ is **stronger** than requiring the existence of partial derivatives so that the Cauchy-Riemann-Equations make sense (Looman-Menchoff Theorem: no need for $\mathcal{C}^\infty$ as $\mathcal{C}^0$ is enough).

Remark. Let $T_z^*\mathbb{C}^n \cong \mathbb{C}^n \cong \mathbb{R}^{2n} \cong \text{span}_{\mathbb{R}}\{dx_i, dy_i\}_{i=1,\dots,n}$. Then the complex basis is given by $T_z^*\mathbb{C}^n \cong \text{span}_{\mathbb{R}}\{dz_i, d\bar{z}_i\}$ where one defines $dz_i := dx_i + idy_i$ and $d\bar{z}_i := dx_i - idy_i$.

Accordingly, one can give the dual basis of $T_z\mathbb{C}^n \cong \mathbb{C}^n \cong \mathbb{R}^{2n}$ with $\partial_{z_i} := \frac{1}{2}(\partial_{x_i} - i\partial_{y_i})$ and $\partial_{\bar{z}_i} := \frac{1}{2}(\partial_{x_i} + i\partial_{y_i})$.

Note that $dz_i, d\bar{z}_i$ and $\partial_{z_i}, \partial_{\bar{z}_i}$ are indeed dual to each other allowing to rewrite the Cauchy-Riemann-Equations in a more compact fashion:

$$\partial_{\bar{z}} f = 0 \tag{4}$$

This follows directly form the definition and rewriting:

$$\partial_{\bar{z}} f = \frac{1}{2}(\partial_x + i\partial_y)(u(x,y) + iv(x,y)) = 0 \;\Leftrightarrow\; \text{Cauchy-Riemann-Equations}$$

Note (Holomorphic ⇔ Complex differentiable)**.** It is possible to prove that a function is holomorphic if and only if it is complex differentiable (in a neighbourhood of each point of its domain). Recall that *complex differentiability* at $z = z_0 \in \mathbb{C}$ means that the limit

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists with $h \in \mathbb{C}$. In particular, complex differentiability implies differentiability, but the converse is not true.

Example 1. The function $f(z) = \bar{z} \in \mathcal{C}^\infty$ is not complex differentiable!

Theorem 1 (Holomorphic ⇔ Analytic)**.** *Let $U \subseteq \mathbb{C}$. Then $F : U \to \mathbb{C}$ is holomorphic if and only if it is analytic.*

Proof. '⇒': Let F be holomorphic and let $B_\varepsilon(z_0) \in U$ s.t. $\partial B_\varepsilon(z_0) =: \mathcal{C}$ with a positive orientation and let $z \in B_\varepsilon(z_0)$. We use the *Cauchy Integral Theorem* stating that in this setting, for all $z \in B_\varepsilon(z_0)$ one has

$$F(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{F(w)}{w-z} dw$$

But then we can proceed as follows:

$$\begin{aligned} F(z) &= \frac{1}{2\pi i}\oint_{\mathcal{C}} dw \frac{F(w)}{(w-z_0)-(z-z_0)} = \frac{1}{2\pi i}\oint_{\mathcal{C}} dw \frac{F(w)}{w-z_0}\left(\frac{1}{1-\frac{z-z_0}{w-z_0}}\right) \\ &= \frac{1}{2\pi i}\oint_{\mathcal{C}} dw \frac{F(w)}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\underbrace{\left(\oint_{\mathcal{C}} dw \frac{F(w)}{(w-z_0)^{n+1}}\right)}_{=:\alpha_n}(z-z_0)^n \end{aligned}$$

Note: it is subtle to prove that the series converges uniformly and absolutely on $\mathcal{C}$ and therefore one can indeed exchange $\oint \leftrightarrow \sum$. To this end, observe that

(i) $\left|\frac{F(w)}{w-z_0}\right| < M$ with $M > 0$ on $\mathcal{C}$,

(ii) For all $w \in \mathcal{C}$ exists an $r \in \mathbb{R}$ such that $\left|\frac{z-z_0}{w-z_0}\right| \leq r < 1$,

implying $\left|\frac{(z-z_0)^n}{(w-z_0)^{n+1}}F(w)\right| \leq Mr^n$ on $\mathcal{C}$. That is, we use the *Weierstrass "M-test"* to prove the convergence.

'$\Leftarrow$': Let F be analytic with Taylor series expansion $F(z) = \sum_{n=0}^{\infty} \alpha_n (z-z_0)^n$ for all $z \in B_\varepsilon(z_0) \subseteq U$. We use the following generalisation of Cauchy's Theorem for smooth functions:

$$F(z) = \frac{1}{2\pi i}\oint_{\partial B} \frac{F(w)}{w-z}dw + \int_B \partial\bar{w}F(w)\frac{dw \wedge d\bar{w}}{w-z}$$

for all $z \in B_\varepsilon(z_0) \subseteq \mathbb{C}$. Then, it is enough to observe the following facts:

(i) The partial sums $\{s_0, s_1, s_2, \dots\}$ where $s_n := \sum_{i=0}^{n} \alpha_i (z-z_0)^i$ satisfy Cauchy's integral formula, $s_n = \frac{1}{2\pi i}\oint_{\partial B} \frac{F_n(w)}{w-z}dw$ near z_0 (because $\partial_{\bar{z}}(z-z_0)^n = 0$).

(ii) By uniform convergence of the series, the same is true for F.

(iii) It follows that $F(z) = \frac{1}{2\pi i}\oint_{\partial B_\varepsilon(z_0)} \frac{F(w)}{w-z}dw$.

(iv) Differentiation with $\partial_{\bar{z}}$ yields $\partial_{\bar{z}}\frac{F(w)}{w-z} = 0$.

Thus, $\partial_{\bar{z}}F = 0$. □

Remark. This proves that the the notions of analytic and holomorphic functions coincide. We will mostly use the latter.

1.2 Fundamental Theorems in One Complex Variable

For a more precise treatment including proofs, see Dr. Kasten's script "Funktionentheorie I" for example.

Theorem 2 (Liouville's Theorem)**.** *Let $F : \mathbb{C} \to \mathbb{C}$ be holomorphic and bounded. Then F is constant.*

Proof. First, we show the following lemma:

Lemma 1. *Let $F : U \to \mathbb{C}$ be holomorphic with $U \subseteq \mathbb{C}$ open and connected (domain). If $F' = 0$, then F is constant on U.*

Proof. We need to show that $F(z_0) = F(z_1)$ for all $z_0, z_1 \in U$. Since U is a domain, it is path-connected. Let $\gamma : [0,1] \to U$ with $\gamma(0) = z_0$ and $\gamma(1) = z_1$ be a path. Then $0 = \int_\gamma F'(w)dw = F(z_1) - F(z_0)$ concluding the proof. □

Back to the proof of Liouville's Theorem: We suppose that $|F(z)| \leq M\ \forall z \in \mathbb{C}$.

(i) To prove that F is constant. we only need to show that $F' = 0$ in $\mathbb{C}$ which is indeed connected.

(ii) Use Cauchy's generalised integral theorem for a path $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) := z_0 + Re^{it}$ with $R > 0$ and $z_0 \in \mathbb{C}$:

$$F'(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{F(w)}{(w-z_0)^2}dw = \frac{1}{2\pi i}\int_0^{2\pi} \frac{F\left(z_0 + Re^{it}\right)}{(Re^{it})^2} iRe^{it}dt = \frac{1}{2\pi R}\int_0^{2\pi} F\left(z_0 + Re^{it}\right)e^{-it}dt$$

(iii) Use that F is bounded:

$$0 \le \left|F'(z_0)\right| \le \frac{1}{2\pi R}\int_0^{2\pi} \left|F\left(z_0 + Re^{it}\right)\right| dt \le \frac{M}{R} \overset{R\to\infty}{\longrightarrow} 0$$

Since $z_0 \in \mathbb{C}$ is arbitrary, it follows $F' = 0$.

Hence, using the previous lemma, we are done. □

Note. This is possibly the most striking difference between real and complex analysis, e.g. $\sin_\mathbb{C} : \mathbb{C} \to \mathbb{C}$ is unbounded!

Note. It implies that there is no (bi-)holomorphic function $\mathbb{C} \to B_1(0)$, i.e. $\mathbb{C} \ncong B_1(0)$.

Theorem 3 (Maximum Principle)**.** *Let $U \subseteq \mathbb{C}$ be a domain and $F : U \to \mathbb{C}$ holomorphic and non-constant. Then $|f|$ has* ***no*** *local maximum in U.*
In particular, if U is bounded and F can be extended to a continuous function $F_\mathcal{C} : \bar{U} \to \mathbb{C}$, then $|f|$ takes its maxima on the boundary ∂U.

Theorem 4 (Identity Theorem)**.** *Let $U \in \mathbb{C}$ be a domain, $f, g : U \to \mathbb{C}$ be holomorphic and let $V \subseteq U$ be an non-empty subset such that $f(z) = g(z)$ on V. Then $f = g$ in U.*

Theorem 5 (Riemann Extension Theorem)**.** *Let $F : B_\varepsilon(z_0) \setminus \{z_0\} \to \mathbb{C}$ be a bounded holomorphic function. Then F can be extended uniquely to a holomorphic function $\tilde{F} : B_\varepsilon(z_0) \to \mathbb{C}$.*

Definition 3 (Bi-holomorphic Function)**.** Let $U, V \in \mathbb{C}$ be open subsets and $f : U \to V$ holomorphic. We call f bi-holomorphic if it is bijective such that f^{-1} is also holomorphic.

Theorem 6 (Little Riemann Mapping Theorem)**.** *Let $U \subsetneq \mathbb{C}$ be a simply connected and open subset in $\mathbb{C}$. Then U is bi-holomorphic to the unit ball $B_1(0) \subseteq \mathbb{C}$.*

Theorem 7 (Residue Theorem)**.** *Let $F : B_\varepsilon(z_0)\setminus\{z_0\} \to \mathbb{C}$ be holomorphic with an isolated singularity in z_0. Then F has a Laurent Series Expansion at z_0*

$$F(z) = \sum_{n=-\infty}^{\infty} \alpha_n (z-z_0)^n \text{ with } \mathrm{Res}_f(z_0) = \alpha_{-1} = \frac{1}{2\pi i}\oint_{|z-z_0|=\varepsilon/2} F(z)dz \tag{5}$$

and $\alpha_n \in \mathbb{C}\ \forall n \in \mathbb{Z}$.

1.3 Several Complex Variables

We now consider the case with more than one complex variable.

Definition 4 (Holomorphic Function ($n > 1$))**.** Let $U \subseteq \mathbb{C}^n$ and let $f : U \to \mathbb{C}$ such that $f \in \mathcal{C}^\infty$. Then f is *holomorphic* if the Cauchy-Riemann-Equations (c.f. 3) with $f = u + iv$ are satisfied for all $z_j = x_j + iy_j$, $j = 1, \dots n$.

Note. Once again, we can rewrite this in a more compact fashion:

$$\left(\begin{cases} \partial_{x_j} u = \partial_{y_j} v \\ \partial_{y_j} u = -\partial_{x_j} v \end{cases} \right) \Leftrightarrow \partial_{\bar{z}_j} f = 0 \tag{6}$$

with $\partial_{\bar{z}_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$.

Note. When $n > 1$, we take *polydisks* as a basis for the topology:

$$B_{\underline{\varepsilon}}\left(\underline{w}\right) = \{\underline{z} \in \mathbb{C}^n : \ |z_j - w_j| < \varepsilon_i \ \forall i\} \tag{7}$$

Theorem	$n = 1$	$n > 1$
Cauchy integral formula	✓	✓
analytic = holomorphic	✓	✓
Liouville's Theorem	✓	✓
Maximum Principle	✓	✓
Identity Theorem	✓	✓
Riemann Extension Theorem	✓	✓
Riemann Mapping Theorem	✓	✗

Table 1: Comparison between $n = 1$ and $n > 1$

Counterexample to the Riemann Mapping Theorem: $\mathbb{C}^2 \supset B_{(1,1)}(0) \subsetneq \mathbb{D}$

Note. Viceversa, there are also theorems which hold true in several variables but not in one variable.

Theorem 8 (Hartog's Extension Theorem)**.** *Let $\underline{\varepsilon} := (\varepsilon_1, \dots, \varepsilon_n)$ and $\underline{\varepsilon}' := (\varepsilon'_1, \dots, \varepsilon'_n)$ with $n > 1$ such that $\varepsilon'_i < \varepsilon_i$ for all $i = 1, \dots, n$. Then any holomorphic map $f : B_{\underline{\varepsilon}}(0) \setminus \overline{B_{\underline{\varepsilon}'}(0)} \to \mathbb{C}$ can uniquely be extended to a holomorphic map $f : B_{\underline{\varepsilon}}(0) \to \mathbb{C}$.*

"Slogan": A holomorphic function in $\mathbb{C}^n \supset U \setminus \{z_0\}$, $z_0 \in \mathbb{C}^n$ extends to a holomorphic function in all U.

Counterexample in $d = 1$: $f(z) = \frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$, but it cannot be extended to a holomorphic function in $\mathbb{C}$!

2 Elements of Sheaf Theory

Local Properties of Holomorphic Functions: a holomorphic function $F : U \to \mathbb{C}$ with a domain $U \subseteq \mathbb{C}$ is determined **completely** by local information.

Remark. This is spelled out precisely in the Identity Theorem (Theorem 4) in complex analysis.

Theorem 9. *Let $U \in \mathbb{C}$ be a domain and let $F, G : U \to \mathbb{C}$ be holomorphic. If $V \subseteq U$ is a non-empty open subset and $F\big|_V = G\big|_V$, then $F = G$ on U.*

Locally, a holomorphic function is represented by its *Taylor Series Expansion*: we now want to study holomorphic functions from this local point of view, i.e. we "forget" the domain of definition of F, but only take into account its "local representations". This leads to the notion of sheaves of holomorphic functions.

Definition 5 (Presheaf)**.** Let X be a topological space. We say that $\mathcal{F}$ is a *presheaf* (of abelian groups) if

1) $X \supseteq U \mapsto \mathcal{F}(U) \in \mathrm{Obj(Ab)}$ ($\mathcal{F}(U)$ is an abelian group for all $U \in X$)

2) For all inclusions $U \subseteq V$, there is a homomorphism of abelian groups, namely the *restriction morphism*:

Definition 6 (Restriction Morphism)**.** This is a homomorphism of abelian groups,
$(V \hookrightarrow U) \mapsto \left(\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)\right)$, such that

1) $\rho^U_U = \mathrm{id}_U$

2) $\rho^V_W \circ \rho^U_V = \rho^U_W$ for $W \subseteq V \subseteq U$

We introduce the following notation: $\rho^U_V(s) =: s\big|_V$.

Note. Usually, one defines $\mathcal{F}(\emptyset) := 0$ (the trivial abelian group), but this is not an axiom.

Note. Elements in $\mathcal{F}(U)$ are called *sections* (of $\mathcal{F}$ over U).

Definition 7 (Sheaf)**.** A presheaf on X is called a *sheaf* (of abelian groups) if it satisfies the following conditions (sometimes called sheaf axioms):

1) **Local Identity:** Let $\{U_j\}$ be open sets in X and $s, t \in \mathcal{F}(U)$ with $U = \bigcup_j U_j$. If $s\big|_{U_j} = t\big|_{U_j}$ for all j, then $s = t$ in U.

2) **Gluing:** Let $\{U_j\}$ be open sets in X with $U = \bigcup_j U_j$. Then for any collection of sections $s_j \in \mathcal{F}(U_j)$ with $s_j\big|_{U_i \cap U_j} = s_i\big|_{U_i \cap U_j}$ for all i, j, there always exists a global section $S \in \mathcal{F}(U)$ such that $s\big|_{U_j} = s_j$ for all j.

Note. By condition 1), the global section in 2) is unique.

Example 2 (Sheaf of Holomorphic Functions)**.** Consider $X = \mathbb{C}^n$. Then
$\mathbb{C}^n \supseteq U \longmapsto \mathcal{O}(U) := \{f : U \to \mathbb{C} \mid f \text{ is holomorphic}\}$ is the *sheaf of holomorphic functions*.

Remark. Although sheaves are defined on open sets, the underlying topological space X consists of points. It is therefore reasonable to try to isolate the behaviour of a sheaf at a point $a \in X$. Conceptually, we do this by looking at a small neighbourhood of the point. If we look at a sufficiently small neighbourhood U_a of a, the behaviour of the sheaf will be the same as the behaviour of the sheaf in $a \in X$.
Problem: No single neighbourhood will be ”small enough”, so we have to take a sort of ”limit” procedure. This leads to the concept of the *direct limit*:

(i) Let $\mathcal{F}$ be a (pre)sheaf on X. For $a \in X$, we consider $\{U_a\}$, the set of all possible open neighbourhoods and we consider the disjoint union $\coprod_{U_a} \mathcal{F}(U_a)$.

(ii) We introduce an equivalence relation on $\coprod_{U_a} \mathcal{F}(U_a)$: let $s \in \mathcal{F}(U_1)$, $t \in \mathcal{F}(U_2)$ with $U_1, U_2 \in \{U_a\}$: Then, define:

$$s \sim_a t :\Longleftrightarrow \exists V \in \{U_a\},\ V \subseteq U_1 \cap U_2 :\ s\big|_V = t\big|_V \tag{8}$$

This means that we consider equivalent the sections that coincide locally.

Definition 8 (Stalk)**.** The *stalk* of a presheaf $\mathcal{F}$ at $a \in X$ is (the abelian group)

$$\mathcal{F}_a := \varinjlim \mathcal{F}(U) := \coprod_{U_a \ni a} \mathcal{F}(U_a)/ \sim_a \tag{9}$$

Definition 9 (Germ)**.** Elements $s_a \in \mathcal{F}_a$ are called *germs* of a section $s \in \mathcal{F}(U_a)$, $a \in U_a$. A germ is represented by a pair: $s_a = (U_a, s)$. In particular, there is a map $\rho_a : \mathcal{F}(U_a) \to \mathcal{F}_a$ such that $s \mapsto s_a := \rho_a(s)$.

Question: For $f \in \mathcal{O}(U)$, holomorphic in $U \in \mathbb{C}^n$, what is the relationship between $\mathcal{O}_{\mathbb{C}^n,a} \ni f_a$ (stalk of f in $a \in U$) and the (convergent) Taylor expansion of f at $a \in U$?

This leads to the following theorem:

Theorem 10. *Let $a \in U \subseteq \mathbb{C}^n$. Then the stalk $\mathcal{O}_{\mathbb{C}^n,a}$ is isomorphic to the algebra of convergent Taylor series at $a \in U$, $\mathbb{C}\{z_1 - a_1, \dots z_n - a_n\}$:*

$$\{f, g \textit{ are holomorphic in } U_a \textit{ and give rise to the same germ } f_a = g_a \textit{ at } a\}$$
$$\Longleftrightarrow$$
$$\{f, g \textit{ have the same Taylor expansion at } a\}$$

Remark. We could restrict to the case $a \in U$ being the origin since translations $\tau_a f(z) := f(z-a)$ induce an isomorphism of algebras $\mathcal{O}_{\mathbb{C}^n,a} \cong \mathcal{O}_{\mathbb{C}^n,0}$.

Remark. $\mathcal{O}_{\mathbb{C}^n,0}$ and $\mathbb{C}\{z_1,\ldots,z_n\}$ are $\mathbb{C}$-algebras, i.e. they satisfy

$$s_a \cdot t_a = (s\cdot t)_a, \quad s_a + t_a = (s+t)_a, \quad \lambda s_a = (\lambda s)_a. \tag{10}$$

We will now focus on some properties of $\mathcal{O}_{\mathbb{C}^n,0} \cong \mathbb{C}\{z_1,\ldots,z_n\}$.

Theorem 11. *The ring $\mathcal{O}_{\mathbb{C}^n,0} \cong \mathbb{C}\{z_1,\ldots z_n\}$ is "very nice". In particular:*

1) $\mathcal{O}_{\mathbb{C}^n,0}$ is local (unique maximal ideal)

2) $\mathcal{O}_{\mathbb{C}^n,0}$ is OFD and Noethernian (follows from the Weierstrass Division Theorem)

Finally, we introduce the notion of meromorphic functions. We recall that in one variable, one has the following definition:

Definition 10 (Meromorphic Function on $U \subseteq \mathbb{C}$)**.** Let $U \subseteq \mathbb{C}$ be open. A function $f : U \to \mathbb{C}$ is *meromorphic* if $f : U \setminus \{p_1,\ldots,p_k\} \to \mathbb{C}$ is holomorphic and f has poles of **finite order** at every point $\{p_1,\ldots,p_k\}$.

One shows that locally $f \sim \frac{g}{h}$ with $\frac{g}{h}$ holomorphic. This generalises to $\mathbb{C}^n$:

Definition 11 (Meromorphic Function on $U \subseteq \mathbb{C}^n$)**.** Let $U \subseteq \mathbb{C}^n$ be open. We say $f : U \to \mathbb{C}$ is *meromorphic* if it is locally a quotient of holomorphic functions, i.e. $f \overset{locally}{\sim} \frac{g}{h}$ with $g,h : U \to \mathbb{C}$ holomorphic.

This means that as a function $f : U \setminus S \to \mathbb{C}$, there exists an open over $\bigcup_i U_i$ of U and holomorphic functions $f_i, g_i : U_i \to \mathbb{C}$ such that $f\big|_{U_i\setminus S} \cdot h_i\big|_{U_i\setminus S} = g_i\big|_{U_i\setminus S}$.

Example 3 (Sheaf of Meromorphic Functions)**.** $U \mapsto K_{\mathbb{C}^n}(U) := \{f : U \to \mathbb{C} \mid f \text{ meromorphic}\}$

Consider the stalk of $K_{\mathbb{C}}^n$ at a point: As it can easily be imagined, the stalk at a point $a \in \mathbb{C}^n$ is such that the following holds:

Theorem 12. *Let $a \in \mathbb{C}^n$. Then $K_{\mathbb{C}^n,a} \cong \mathbb{C}_{Laurent}\{x_1 - a_1,\ldots,x_n - a_n\}$ (convergent Laurent Series at $a \in \mathbb{C}^n$).*

Note. $K_{\mathbb{C}^n,a}$ is a **field** (no ideal except of itself and the trivial one) and indeed $K_{\mathbb{C}^n,a}$ is the *field of fractions* of the integral domain $\mathcal{O}_{\mathbb{C}^n,a}$, that is:

$$K_{\mathbb{C}^n,a} = \mathrm{Frac}\,(\mathcal{O}_{\mathbb{C}^n,a}) \cong \mathbb{C}_{Laurent}\{x_1 - a_1,\ldots,x_n - a_n\} \tag{11}$$

This "justifies" that $f = \frac{g}{h}$ locally.

3 Complex Manifolds

3.1 Basic Defintions

We let X be a *topological manifold* (i.e. it has the Hausdorff property and it is locally homeomorphic to an open set $V \in \mathbb{R}^n$).

Definition 12 (Complex Chart)**.** A local *complex chart* (U, φ) of X is an open set $U \subseteq X$ with a homeomorphism $\varphi : U \to \varphi(U) \subseteq \mathbb{C}^n$ (where $\mathbb{C}^n \cong \mathbb{R}^{2n}$).

Compatibility: Let $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) be two complex charts. We say they are compatible if the transition functions

$$\varphi_{\beta\alpha} := \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha\big(\underbrace{U_\alpha \cap U_\beta}_{\subseteq \mathbb{C}^n}\big) \to \varphi_\beta\big(\underbrace{U_\alpha \cap U_\beta}_{\subseteq \mathbb{C}^n}\big) \tag{12}$$

are **holomorphic**.

Note. Observe that $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ is holomorphic too.

Definition 13 (Holomorphic Atlas)**.** A *holomorphic atlas* of a space X is a collection of local charts $\mathcal{A} := \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ such that $X = \bigcup_\alpha U_\alpha$ and all the transition functions $\varphi_{\alpha\beta}$ are **bi-holomorphic** for all α, β. In this way, each pair of charts is compatible.

Definition 14 (Holomorphic Structure)**.** A *holomorphic structure* on X is a **maximal** holomorphic atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$. Maximal means that if (U, φ) is a chart and compatible with $(U_\alpha, \varphi_\alpha)$ for all $\alpha \in I$, then $(U, \varphi) \in \mathcal{A}$.

Definition 15 (Complex Manifold)**.** A *complex manifold* is a topological manifold together with a holomorphic structure.

Note. A holomorphic atlas $\mathcal{B} = \{(U_\beta, \varphi_\beta)\}_{\beta \in J}$ determines a unique maximal atlas $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{A}$. As such it determines the complex manifold.

The atlas is given by $\mathcal{A} = \{(U, \varphi) : (U, \varphi) \text{ is compatible with } (U_\beta, \varphi_\beta)\ \forall \beta \in J\}$.

Remark (Complex Manifolds and Real Manifolds)**.** Given a complex manifold X, we can think about it without its holomorphic structure:

If $\dim_{\mathbb{C}} X = n$, then X defines a *differentiable manifold* X_o with $\dim_{\mathbb{R}} X_o = 2n$. A complex chart (U, φ) gives rise to a real chart $(U, \tilde{\varphi})$ via

$$\varphi = (z_1, \ldots, z_n) \longleftrightarrow \tilde{\varphi} = (x_1, \ldots, x_n, y_1, \ldots, y_n) \tag{13}$$

with $z_j = x_j + iy_j$ for all $j = 1, \ldots, n$.

Theorem 13 (Complex Manifolds and Orientability)**.** *Consider a complex manifold X as a real manifold X_o. Then X_o is **orientable**.*

Proof. Any transition function $\varphi_{\beta\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic and so is its inverse. We have that $\det(J_\mathbb{R}\varphi_{\beta\alpha}) = |\det(J_\mathbb{C}\varphi_{\beta\alpha})|^2 > 0$ (exercise!). Notice that it is non-zero as $\varphi_{\beta\alpha}$ has an inverse. Now $J_\mathbb{R}\varphi_{\beta\alpha}$ is the jacobian of the transition functions $\tilde{\varphi}_{\beta\alpha}$ on X_o. Then every transition function has positive determinant: it follows that X_o is equipped with a positive atlas, hence it is (positively) oriented. □

Consequence: Not every (even dimensional) differentiable manifold X_o^{2n} can be seen as the underlying differentiable manifold of a complex manifold.

Definition 16 (Holomorphic Functions)**.** Let $U \subseteq X$ be an open set. Then $f : U \to \mathbb{C}$ is *holomorphic* if for charts $(U_\alpha, \varphi_\alpha) \subseteq \mathcal{A}$ with $U_\alpha \cap U \neq \emptyset$

$$f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U) \to \mathbb{C} \tag{14}$$

is holomorphic.

Sheaf of Holomorphic Functions:

$$X \supseteq U \longmapsto \mathcal{O}_X(U) := \{f : U \to \mathbb{C} \mid f \text{ is holomorphic}\} \tag{15}$$

Note. It follows from the definition that using a chart (U, φ) with $\varphi(\tilde{x}) = 0$ for $\tilde{x} \in U$, then $\mathcal{O}_{X,x} \cong \mathcal{O}_{\mathbb{C}^n,0}$. **Stalks coincide with those of $\mathbb{C}^n$.**

Remark. Let $(U, \varphi = (z_1, \ldots, z_n))$ a complex chart with $x \in U$, $\varphi(x) = 0$ and let $f : U \to \mathbb{C}$ be holomorphic. Then we have

$$\left(f \circ \varphi^{-1}\right)(w) = \sum_{k_1,\ldots,k_n=0}^{\infty} \alpha_{k_1,\ldots,k_n} w_1^{k_1} \ldots w_n^{k_n} \tag{16}$$

with $x \in U$ and $\varphi(x) = w$. This means that $f(x) = \left(f \circ \varphi^{-1})\right)(\varphi(x)) = \sum_{\underline{k}}^{\infty} \alpha_{k_1,\ldots,k_n} (\underbrace{\varphi_1(x)}_{z_1(x)})^{k_1} \ldots (\underbrace{\varphi_n(x)}_{z_n(x)})^{k_n}$.

Hence: $f = \sum_{\underline{k}}^{\infty} \alpha_{\underline{k}} z_1^{k_1} \ldots z_n^{k_n}$, the Taylor expansion at a point.

Definition 17 (Holomorphic Map $X \to Y$)**.** A map $f : X \to Y$ between complex manifolds is *holomorphic* if $\psi_\beta \circ f \circ \phi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap f^{-1}(V_\beta)) \to \varphi_\beta(U_\beta)$ is holomorphic for all charts $(U_\alpha, \varphi_\alpha)$ of X and (V_β, ψ_β) of Y. It is sufficient to verify in one atlas of X and Y.
We say that the manifolds are *isomorphic*, $X \cong Y$, if there exists a holomorphic homeomorphism $X \to Y$. Note that f^{-1} is holomorphic as well.

We now come to a crucial result:

Theorem 14 (Global Sections of $\mathcal{O}_X$)**.** *Let X be a compact and connected complex manifold. Then $\mathcal{O}_X(X) \cong \mathbb{C}$.*

Proof. Let $f : X \to \mathbb{C}$ be holomorphic. Then, f is continuous and so is $|f|$. It follows that $|f|$ has a maximum at some $x \in X$ since X is compact. But, if (U, φ) is a chart with $x \in U$, then $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ is locally constant by the Maximum Principle (theorem 3). Finally, since X is connected, the identity principle (theorem 4) implies that f has to be constant. □

Comment: There are **no** non-constant holomorphic functions and as such there are **no** embeddings in $\mathbb{C}^n$. Usually, compactness makes life easier. Instead, it tells us here that we are allowed to deal with holomorphic functions because they are all constant.

3.2 Examples

Complex Projective Space

As usual, we define the *complex projective space*

$$\mathbb{P}^n \ (:= \mathbb{CP}^n) := \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \sim, \tag{17}$$

where $u \sim v \Leftrightarrow u = tv$ for $t \in \mathbb{C}^\times$ and $u, v \in \mathbb{C}^{n+1} \setminus \{0\}$. Note that there is an action (**proper** and **free**) of $\mathbb{C}^\times$ on $\mathbb{C}^{n+1} \setminus \{0\}$; the quotient by this action is $\mathbb{P}^n$. In other words, we let

$$\begin{aligned} \pi : \mathbb{C}^{n+1} \setminus \{0\} &\longrightarrow \mathbb{P}^n \\ (x_0, \dots, x_n) &\longmapsto [x_0 : x_1 : \cdots : x_n] \end{aligned} \tag{18}$$

This is the quotient map and $[x_0 : \cdots : x_n]$ are called *homogeneous coordinates.*

Topology: $\mathbb{P}^n$ has the quotient topology: $U \subseteq \mathbb{P}^n$ is open if $\pi^{-1}(U) \subseteq \mathbb{C}^{n+1} \setminus \{0\}$ is open.

The usual atlas $\mathcal{A}_{\mathbb{P}^n} = \{(U_j, \varphi_j\})_{j=0,\dots,n}$ is given by

$$\begin{aligned} \varphi_j : U_j &\longrightarrow \mathbb{C}^n \\ [x_0 : \cdots : x_n] &\longmapsto \left(\frac{x_0}{x_j}, \dots, \frac{\widehat{x_j}}{x_j}, \dots, \frac{x_n}{x_j}\right) \end{aligned} \tag{19}$$

with $U_j = \{[x_0 : \cdots : x_n] : x_j \neq 0\}$. Notice that the inverse map is given by $\varphi_j^{-1} : (x_1, \dots, x_n) \mapsto [x_1 : \cdots : 1 : \cdots : x_n]$.

Compatibility: As an easier example, we verify the compatibility between (U_0, φ_0) and (U_1, φ_1). The transition functions yield:

$$\begin{aligned} \varphi_0 \circ \varphi_1^{-1} : \varphi_1 (U_0 \cap U_1) &\longrightarrow U_0 \cap U_1 \longrightarrow \varphi_0 (U_0 \cap U_1) \\ (x_0, x_2, \dots, x_n) &\longmapsto [x_0 : 1 : \cdots : x_n] \longmapsto \left(\frac{1}{x_0}, \frac{x_2}{x_0}, \dots, \frac{x_n}{x_0}\right) \end{aligned}$$

Note that $\varphi_0 \circ \varphi_1^{-1} =: \varphi_{01} : \varphi_1(U_0 \cap U_1) \to \varphi_0(U_0 \cap U_1)$ is indeed holomorphic.

Lemma 2. *$\mathbb{P}^n$ is compact for any n.*

Proof. We let $S^{2n+1} = \{u \in \mathbb{C}^{n+1} : \|u\| = \sqrt{\sum_i |u_j|^2} = 1\}$. We know that S^{2n+1} is compact and we can observe that $\pi\big|_{S^{2n+1}} : S^{2n+1} \to \mathbb{P}^n$ is surjective. Indeed, if $p = \pi(u) \in \mathbb{P}^n$, there exists a $t \in \mathbb{C}^\times$ such that $\|tu\| = 1$ which implies $tU \in S^{2n+1}$ and $\pi(tu) = \pi(u) = p$. Now, the map π is continuous and maps compact sets to compact sets. □

Note. $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is holomorphic, hence continuous. Indeed, let us check this using atlases $\{(\mathbb{C}^{n+1} \setminus \{0\}, \mathrm{id}_{\mathbb{C}^{n+1}\setminus\{0\}})\}$ on $\mathbb{C}^{n+1} \setminus \{0\}$ and the standard atlas $\{(U_j, \varphi_j)\}_j$ on $\mathbb{P}^n$. We look at $j = 0$:

$$\pi \rightsquigarrow \varphi_0 \circ \pi \circ \mathrm{id}_{\mathbb{C}^{n+1}\setminus\{0\}}(z_0, \ldots, z_n) = \varphi_0\left([z_0 : \cdots : z_n]\right) = \left(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right)$$

The map is clearly holomorphic on $\pi^{-1}(U_0) \subseteq \mathbb{C}^{n+1} \setminus \{0\}$.

Remark (Sheaves on $\mathbb{P}^n$)**.** First, we define the *sheaf of regular functions* on $\mathbb{P}^n$:

$$U \longmapsto \mathcal{O}_{\mathbb{P}^n}(U) := \left\{ f \in \mathcal{O}_{\mathbb{C}^{n+1}\setminus\{0\}}\left(\pi^{-1}(U)\right) : f(\lambda x) = f(x)\ \forall x \in \pi^{-1}(U), \lambda \in \mathbb{C}^\times \right\} \tag{20}$$

Exercise: Let $(x_0, x_1) \in \mathbb{C}^n$ with $x_0 \neq 0$. Show: Then, $F = \frac{x_1}{x_0} \in \mathcal{O}_{\mathbb{P}^1}(U_0)$.

- Notice that $f \in \mathcal{O}_{\mathbb{C}^{n+1}\setminus\{0\}}(\pi^{-1}(U))$. The corresponding regular function F on U is well-defined as $F(\pi(x)) = f(x)$.
- Notice that $\mathcal{O}_{\mathbb{P}^n}$ is a sheaf of rings.

The sheaves $\mathcal{O}_{\mathbb{P}^n}(k)$: Let $k \in \mathbb{Z}$ and we define:

$$U \longmapsto \mathcal{O}_{\mathbb{P}^n}(k)(U) := \left\{ G \in \mathcal{O}_{\mathbb{C}^{n+1}\setminus\{0\}}\left(\pi^{-1}(U)\right) : G(\lambda x) = \lambda^k G(x)\ \forall x \in \pi^{-1}(U), \lambda \in \mathbb{C}^\times \right\} \tag{21}$$

This sheaf has the following properties:

- It is an abelian group with $(G + H)(x) = G(x) + H(x)$.
- It is a $\mathcal{O}_{\mathbb{P}^n}(U)$-module with $(fG)(x) = f(\pi(x))G(x)$ for $f \in \mathcal{O}_{\mathbb{P}^n}(U)$.
- It is locally free (of rank 1), i.e. for all $U \subseteq U_j$ one has an isomorphism

$$U \longmapsto \begin{cases} \mathcal{O}_{\mathbb{P}^n}(k)(U) \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}^n}(U) \\ G \longmapsto x_j^{-k} G \end{cases} \tag{22}$$

  It follows from this that the product map $\mathcal{O}_{\mathbb{P}^n}(k) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(l) \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}^n}(l+k)$ given by $G \otimes H \longmapsto GH$ is an isomorphism.

Complex Tori

Definition 18 (Lattice)**.** Let $\mathbb{C}^n$ be seen as a $\mathbb{R}$-vector space and consider $2n$ linearly independent vectors $\{w_1, \dots, w_{2n}\}$ over $\mathbb{R}$, that is $\mathbb{C}^n = \mathbb{R}w_1 \oplus \cdots \oplus \mathbb{R}w_{2n}$. A *lattice* in $\mathbb{C}^n$ is defined as the subset

$$\Lambda := \left\{ \lambda \in \mathbb{C}^n : \lambda = \sum_{i=1}^{2n} k_i w_i, \; k_i \in \mathbb{Z} \right\} \tag{23}$$

Note. $\Lambda \subseteq \mathbb{C}^n$ is an additive subgroup of $\mathbb{C}^n$ and it is isomorphic to $\mathbb{Z}^{2n}$.

Definition 19 (Complex Torus)**.** A *complex torus* is defined as the quotient $\mathbb{C}^n/\Lambda =: A^{(n)}$

Remark. As a group, we have $\mathbb{C}^n/\Lambda \cong \mathbb{R}^{2n}/\mathbb{Z}^{2n} \cong \left(\mathbb{R}/\mathbb{Z}\right)^{2n} \cong (S^1)^{2n}$. This explains the name "torus".

Topology: It is worth observing that $A^{(n)}$ can also be seen as a quotient with an equivalence relation, i.e. $A^{(n)} = \mathbb{C}^n/\sim$, where $z \sim w \Leftrightarrow z - w \in \Lambda$. It follows that $A^{(n)}$ is a topological space with the quotient topology, moreover it is Hausdorff.

- $\pi : \mathbb{C}^n \to \mathbb{C}^n/\Lambda$ is open: Indeed, let $V \subseteq \mathbb{C}^n$ be open and consider $\pi(V)$. One has that $\pi(V)$ is open if $\pi^{-1}(\pi(V))$ (the "saturation" of V) is open, but $\pi^{-1}(\pi(V)) = \bigsqcup_{\lambda \in \Lambda}(V + \lambda)$ where the right hand side is open because it is an infinite union of (translated) open sets in $\mathbb{C}^n$.
- $A^{(n)}$ is compact: We have $A^{(n)} = \pi(\hat{\Lambda})$ with $\hat{\Lambda} = \{\sum_i t_i w_i, t \in [0,1]\}$. But since $\hat{Lambda}$ is compact and π is continuous, $A^{(n)}$ is compact. (Notice $A^{(n)} \cong (S^1)^{2n}$)

Charts and Atlas: For $x \in A^{(n)}$, let us consider some $z \in \mathbb{C}^n$ such that $\pi(z) = x$.

(i) Choose a neighbourhood $V \subseteq \mathbb{C}^n$ for $z \in \mathbb{C}^n$ such that $\pi_V := \pi\big|_V \overset{\cong}{\to} \pi(V)$ is a bijection. Notice that this is always possible, e.g. using $V = \{z + \sum_{i=1}^{2n} t_i w_i : |t_i| < \frac{1}{2} \; \forall i = 1, \dots, 2n\}$

(ii) Then one has in particular that if $z, z' \in V$, $z \neq z' + \Lambda$ so that $z \not\sim z'$ unless $z = z'$. This means that $\pi_V V \to \pi(V)$ is injective.

(iii) Since π_V is open and injective, it is a homoemorphism.

Thus, $(\pi(V), \pi_V^{-1})$ is a complex chart for $x \in A^{(n)}$.

Compatibility: Let $V, W \subseteq \mathbb{C}^n$, $V \cap W \neq \emptyset$ and $\pi(V), \pi(W) \subseteq A^{(n)}$. Then, we have $\pi_W^{-1} \circ (\pi_V^{-1})^{-1} : \pi_V^{-1}(\pi(V) \cap \pi(W)) \to \pi_W^{-1}(\pi(V) \cap \pi(W))$. Consider a point $z \in \pi_V^{-1}(\pi(V) \cup \pi(W))$ with $z' = \pi_W^{-1} \circ (\pi_V^{-1})^{-1}(z)$ and apply $\pi_W : W \overset{\cong}{\longrightarrow} \pi(W)$ to find $\pi_V(z) = \pi_W(z')$. This implies $\pi(z) = \pi(z')$, so there exists a $\lambda \in \Lambda$ with $z' = z + \lambda$. Hence: $\pi_W^{-1} \circ (\pi_V^{-1})^{-1}(z) = z + \lambda$.

Conclusion: Transition functions are translations by elements in Λ for any choice of V and W. In particular, they are holomorphic and $A^{(n)}$ is a complex manifold.

Note. $\pi : \mathbb{C}^n \to A^{(n)}$ is holomorphic. Indeed, restricting π to the sets V where it is a bijection yields $\pi_V^{-1} \circ \pi_V \circ \mathrm{id}_{\mathbb{C}^n} = \mathrm{id}_{\mathbb{C}^n}$.

Remark (Sheaves on $A^{(n)}$)**.** The sheaf of regular functions on $A^{(n)}$ is given by

$$U \longmapsto \mathcal{O}_{A^{(n)}}(U) = \left\{ f \in \mathcal{O}_{\mathbb{C}^n}\left(\pi^{-1}(U)\right) \to \mathbb{C} : f(z+\lambda) = f(z)\ \forall z \in \pi^{-1}(U),\ \forall \lambda \in \Lambda \right\} \tag{24}$$

The relation with $F : U \subseteq A^{(n)} \to \mathbb{C}$ is given by $F(\pi(z)) = f(z)$. Sometimes, these functions are called Λ-*periodic functions.*

3.3 Complex Submanifolds

Definition 20 (Complex Submanifold)**.** A *complex submanifold* of a complex manifold X with $\dim_{\mathbb{C}} X = n$ is a subset $Y \subseteq X$ such that $\forall a \in Y$ there exists a local complex chart $(U, \varphi = (z_1, \ldots, z_n))$ of X, called the *preferred chart*, with $\varphi(a) = 0$ and

$$\varphi(U \cap Y) = \{u \in \varphi(U) \subseteq \mathbb{C}^n : u_{k+1} = \cdots = u_n = 0\} \tag{25}$$

Alternatively, there exists a holomorphic atlas for X, $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ such that $\varphi_\alpha\big|_{U_\alpha \cap Y} : U_\alpha \cap Y \xrightarrow{\cong} \varphi_\alpha(U_\alpha) \cap \mathbb{C}^k$ where $\mathbb{C}^k \hookrightarrow \mathbb{C}^n$, $(z_1, \ldots, z_k) \mapsto (z_1, \ldots, z_n, 0, \ldots, 0)$.
Note that $\mathrm{codim}_X Y. = \dim X - \dim Y = n - k$.

Note. A complex submanifold is itself a complex manifold of dimension k. If (U, φ) is a preferred chart, one obtains a complex chart as above by using $(U \cap Y, \varphi|_{U \cap Y})$. Note that the compatibility of these charts follows from those of X.

Note. We now want to provide methods to obtain complex submanifolds and we will see that, on a very general ground, there are two such possibilities to do so:

1) The preimage of a point via a "sufficiently regular" map is a submanifold.

2) Under strong conditions, the image $\varphi(X)$ of a map $\varphi : X \to Y$ is an embedded submanifold of Y. This means that $\varphi(X) \subseteq Y$ in some "non-singular" way.

Theorem 15 (Preimage Manifold)**.** *Let $\varphi : X^{(n)} \to Y^{(m)}$ be a holomorphic map between complex manifolds with $n > m$ and let $b \in \varphi(X^n) \subseteq Y^{(m)}$ be such that the rank of φ is maximal, i.e.* $\mathrm{rank}(J_{\mathbb{C}}\varphi) = m$ *for all $a \in \varphi^{-1}(b)$. Then $\varphi^{-1}(b)$ is a complex submanifold of dimension $n - m$.*

Theorem 16 (Embedded Manifold)**.** *Let $\varphi : Y \hookrightarrow X$ be an injective holomorphic map with $m = \dim Y, \dim X = n$ and $m \leq n$ such that φ has maximal rank m on all Y. If Y is compact, then $\varphi(Y)$ is a submanifold of X and $\varphi : Y \to \varphi(Y) \subseteq X$ is a holomorphic map. We say that $\varphi(Y)$ is isomorphic to Y and φ is an embedding of complex manifolds.*

We will now look at some examples which are characterised by the fact that the submanifold is embedded into some $\mathbb{P}^n$:

Example 4 (Veronese Map)**.** Consider

$$\begin{aligned} \varphi_d : \mathbb{P}^n &\longrightarrow \mathbb{P}^m \\ [x_0 : \cdots : x_m] &\longmapsto [x_0^d : x_0^{d-1}x_1 : \cdots : x_n] \end{aligned} \tag{26}$$

with $m := \binom{n+d}{n} - 1$. It maps $[x_0 : \cdots : x_n]$ in all possible monomials in d variables of degree d. The case $n = 1$ corresponding to $\varphi_d(\mathbb{P}^1)$ is called *rational normal curve.*

Example: The *twisted cubic curve*:

$$\begin{aligned} \varphi_3 : \mathbb{P}^1 &\longrightarrow \mathbb{P}^3 \\ [s, t] &\longmapsto [s^3 : s^2t : st^2 : t^3] \end{aligned} \tag{27}$$

Example 5 (Segre Map)**.** Consider

$$\begin{aligned} \sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m &\longrightarrow \mathbb{P}^{(n+1)(m+1)-1} \\ ([x_0 : \cdots : x_n], [y_0 : \cdots : y_m]) &\longmapsto [x_0y_0 : x_0y_1 : \cdots : x_iy_{j-1} : x_iy_j : \cdots : x_ny_m] \end{aligned} \tag{28}$$

Example: The *quadric* (in $\mathbb{P}^3$):

$$\begin{aligned} \sigma_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^3 \\ ([s, t], [u, v]) &\longmapsto [su : sv : tu : tv] \end{aligned} \tag{29}$$

Example 6 (Complete Intersections)**.** We let f be a homogeneous polynomial of degree d, i.e. $f(tx) = t^d f(x)\ \forall t \in \mathbb{C}$. Then:

$$\frac{d}{dt}f(tx) = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(tx_i)x_i = d \cdot t^{d-1}f(x) \overset{t=1}{\Longrightarrow} \sum_{i=0}^{n} x_i \frac{\partial f}{\partial x_i}(x) = d \cdot f,$$

the *Euler equation.*

Theorem 17 (Complete Intersections)**.** *Let $f_1, \ldots, f_m \in \mathbb{C}[x_0, \ldots, x_n]$ be homogeneous polynomials of degree d_j, $j = 1, \ldots, m$ for some $m < n$. We have the projective algebraic set*

$$Y := \{x \in \mathbb{P}^n : f_1(x) = \cdots = f_m(x) = 0\}. \tag{30}$$

Then, if $\operatorname{rank}(\partial_{x_k} f_j) = m\ \forall x \in Y,\ \forall k = 0, \ldots, n,\ \forall j = 1, \ldots, m$, *$Y$ is a complex submanifold of $\mathbb{P}^n$ which is compact of dimension $n - m$. We call Y a complete intersection.*

Note. This realises complex (sub)manifolds as the zero locus of homogeneous polynomials in $\mathbb{P}^n$.

Example 7 (Conic in $\mathbb{P}^2$ as a Complete Intersection)**.** Consider

$$Y := \left\{[x_0 : x_1 : x_2] : P(x) = x_0x_2 - x_1^2 = 0\right\} \tag{31}$$

with a homogeneous polynomial P of degree 2. We want to show that Y is a complex submanifold of $\mathbb{P}^2$ of dimension 1 as a complete intersection of degree 2 in $\mathbb{P}^2$. This means that we need to show that $\mathrm{rank}(\partial_x P) = 1$:

$$v = \left(\frac{\partial P}{\partial x_0}, \frac{\partial P}{\partial x_1}, \frac{\partial P}{\partial x_2}\right) = (x_2, -2x_1, x_0) \stackrel{!}{=} 0 \Leftrightarrow x_0 = x_1 = x_2 = 0$$

As $0 \notin \mathbb{P}^2$, it follows that $\mathrm{rank}(\partial_x P) = 1$.

Complete Intersection and Veronese map $\varphi_2(\mathbb{P}^1)$: (degree 2 *rational normal curve*) Actually, the above complete intersection is isomorphic to the Veronese variety

$$\begin{aligned} \varphi_2 : \mathbb{P}^1 &\longrightarrow \mathbb{P}^2 \\ [s:t] &\longmapsto [s^2 : st : t^2] \end{aligned} \tag{32}$$

- $\varphi_2(\mathbb{P}^2) \subseteq Y$: Indeed, $P(s^2, st, t^2) = s^2t^2 - s^2 - t^2 = 0$.

- $Y \subseteq \varphi_2(\mathbb{P}^2)$: Consider $[x_0 : x_1 : x_2] \in Y$. We suppose $x_0 \neq 0$ so that we assume $[1 : x_1 : x_2]$: we have $x_2 = x_1^2$ so that $x = [1 : x_1 : x_1^2]$, but $[1 : x_1 : x_1^2] = \varphi_2([1 : x_1])$. Now, suppose $x_0 = 0$ which implies $x_1 = 0$ so that one has $x = [0 : 0 : x_2]$. But then: $[0 : 0 : x_2] = [0 : 0 : 1] = \varphi_2([0 : 1])$.

Thus: $Y \cong \mathbb{P}^1$.

Example 8 (Quadrics in $\mathbb{P}^3$ and Segre Map)**.** Similarly as above, one can show that

$$\sigma_{1,1}\left(\mathbb{P}^1 \times \mathbb{P}^1\right) \cong \left\{ x \in \mathbb{P}^3 : \det \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} = 0 \right\} \cong \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3 \tag{33}$$

N.B.: On the other hand, $\varphi_3(\mathbb{P}^1)$ (*twisted cubic curve*) is **not** a complete intersection!

Theorem 18. *1) Any (smooth) conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.*

2) Any (smooth) quadric in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

3.4 Submanifolds and Sheaves: Ideal Sheaves

Any sheaf $\mathcal{F}$ on $Y \stackrel{i}{\hookrightarrow} X$ can be considered as a sheaf on X: The push-forward or *direct image sheaf*:

$$\begin{aligned} i_* : \underline{\mathrm{Sh}}(Y) &\longrightarrow \underline{\mathrm{Sh}}(X) \\ \mathcal{F} &\longmapsto i_*\mathcal{F} \end{aligned} \tag{34}$$

where we define $X \supseteq U \mapsto i_* \mathcal{F}(U) := \mathcal{F}(i^{-1}(U))$ ($i^{-1}(U)$ is open in Y). If $\varphi : \mathcal{F} \to \mathcal{G}$ on Y, then $i_*(\varphi) : i_*\mathcal{F} \to i_*\mathcal{G}$ is a sheaf morphism on X. This means that a $\mathcal{O}_Y$-sheaf $\mathcal{F}$ can be looked at as a $\mathcal{O}_X$-sheaf supported on Y.

Further, the restriction of holomorphic functions yields a natural surjection: $i^{\#} : \mathcal{O}_X \to i\mathcal{O}_Y$ (this is seen as a sheaf on X, it is simply denoted as $\mathcal{O}_Y$).

It follows that one has a short exact sequence of sheaves (on X), the *structure sheaf sequence*:

$$0 \longrightarrow I_Y \longrightarrow \mathcal{O}_X \xrightarrow{i^{\#}} \mathcal{O}_Y \longrightarrow 0 \tag{35}$$

The sheaf I_Y is called *ideal sheaf*:

$$X \supseteq U \longmapsto I_Y(U) := \{f : U \to \mathbb{C} : f \text{ is holomorphic and vanishing on } Y \subseteq X\} \tag{36}$$

This is the way one looks at submanifolds on a sheaf-theoretical ground.

4 Vector Bundles and Line Bundles

4.1 Bundles, Sections and Adjunction

Definition 21 (Holomorphic Vector Bundle)**.** A *holomorphic vector bundle* of rank r on a complex manifold X is a complex manifold E together with a surjective holomorphic map $\pi : E \to X$ such that

1) Each *fibre* $E_x := \pi^{-1}(x)$ is a complex vector space of dimension n.

2) There exists an open covering $X = \bigcup_{\alpha \in I} U_\alpha$ and a family of bi-holomorphisms called *local trivialisations*

$$\psi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{C}^r \tag{37}$$

such that they are linear isomorphisms on the fibers and the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\psi_\alpha} & U_\alpha \times \mathbb{C}^r \\ & {\scriptstyle\pi}\searrow & \downarrow {\scriptstyle p_1} \\ & & U_\alpha \end{array}$$

Remark (Transition Functions)**.** We can look at the transition functions between the local trivialisations:

$$\begin{aligned} \psi_\alpha \circ \psi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r &\longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{C}^r \\ (x, v) &\longmapsto (x, g_{\alpha\beta}(x)v) \end{aligned} \tag{38}$$

Remark (Important!)**.** The map $x \mapsto g_{\alpha\beta}(x)$ is holomorphic and one has $g_{\alpha\beta}(x) \in \mathrm{GL}(r, \mathbb{C})$, together with

1) $g_{\alpha\alpha} = \mathrm{id}_r$

2) $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$

3) $\underbrace{g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma}}_{\text{cocycle conditions in } U_\alpha \cap U_\beta \cap U_\gamma}$ (because $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = \mathrm{id}_r$).

Note. The data $\{U_\alpha\}_{\alpha \in I}, \{g_{\alpha\beta}\}_{\alpha,\beta \in I})$ determines the vector bundle "uniquely", $E \longleftrightarrow \{U_\alpha, g_{\alpha\beta}\}$.

Proof of the remark 4.1. For all $x \in U_\alpha \cap U_\beta$ there exists a $g_{\alpha\beta}(x) \in \mathrm{GL}(r, \mathbb{C})$ such that $\psi_{\alpha\beta}(x, v) = (x, g_{\alpha\beta}(x)v)$ since $\psi_{\alpha\beta}|_{\pi^{-1}(x)}$ is an isomorphism of vector spaces and fibre preserving. On $U_\alpha \cap U_\beta \cap U_\gamma$, we have:

$$\begin{cases} \psi_{\alpha\beta} \circ \psi_{\beta\gamma}(x, v) = \psi_{\alpha\beta}\left(x, g_{\beta\gamma}(x)v\right) = (x, \underbrace{g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x)}_{\text{matrix multiplication}} \; v) \\ \psi_{\alpha\gamma}(x, v) = (x, g_{\alpha\gamma}(x)v) \end{cases}$$

Since we have $\psi_{\alpha\beta} \circ \psi_{\beta\gamma} = \psi_\alpha \circ (\psi_\beta^{-1} \circ \psi_\beta) \circ \psi_\gamma^{-1} = \psi_\alpha \circ \psi_\gamma^{-1} \equiv \psi_{\alpha\gamma}$, this implies $g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}$. In addition, we can conclude:

- $\alpha = \beta = \gamma$: $g_{\alpha\alpha} \circ g_{\alpha\alpha} = g_{\alpha\alpha} \implies g_{\alpha\alpha} = \mathrm{id}$.
- $\alpha = \gamma$: $g_{\alpha\beta} \circ g_{\beta\alpha} = g_{\alpha\alpha} \implies g_{\alpha\beta} \circ g_{\beta\alpha} = \mathrm{id}$.

□

Definition 22 (Holomorphic Section)**.** A *holomorphic section* is a holomorphic map $s : X \to E$ such that it preserves fibres of E (i.e. $\pi \circ s = \mathrm{id}_X$). Usually, it is only defined locally: $s : U \to E$.

Note (Zero Section)**.** There is always at least one global section, the *zero section*: $x \mapsto 0 \in E_x \; \forall x \in X$.

Note. For all open sets $U \subseteq X$ the space $\Gamma(U, E) := \{s : U \to E \text{ holomorphic}\}$ is naturally a complex vector space.

Local Representation of Sections: Consider $E \longleftrightarrow (U_\alpha, g_{\alpha\beta})$ and $s \in \Gamma(U, E)$. Then

$$U_\alpha \cap U \ni x \overset{s}{\longmapsto} s(x) \overset{\psi_\alpha}{\longmapsto} (x, s_\alpha(x)) \in (U_\alpha \cap U, \mathbb{C}^r)\,, \tag{39}$$

note that $s_\alpha : U_\alpha \cap U \to \mathbb{C}^r$ is holomorphic! Under a change of trivialisation, one has:

$$\begin{cases} \psi_{\alpha\beta}\left(x, s_\beta(x)\right) = \psi_\alpha \circ \psi_\beta^{-1} \circ \psi_\beta\left(s(x)\right) = \psi_\alpha\left(s(x)\right) = \left(x, s_\alpha(x)\right) \\ \psi_{\alpha\beta}\left(x, s_\beta(x)\right) = \left(x, g_{\alpha\beta}(x) s_\beta(x)\right) \end{cases}$$

Thus: $s_\alpha(x) = g_{\alpha\beta}(x) s_\beta(x)$.

Remark. Conversely, going in the other direction, a collection of local sections $s_\alpha : U_\alpha \to \mathbb{C}^r$ determines uniquely a "global" section $s : U \to E$. Indeed, $s(x) = \psi_\alpha^{-1}(x, s_\alpha(x))$ and this is independent of the chart:

$$\psi_\alpha^{-1}(x, s_\alpha(x)) = \psi_\alpha^{-1}\left(x, g_{\alpha\beta}(x)s(x)\right) = \psi_\alpha^{-1} \circ \left(\psi_\alpha \circ \psi_\beta^{-1}(x, s_\beta(x)\right) = \psi_\beta^{-1}(x, s_\beta(x))$$

Local Description of $S : U \to E$:

$$s \longleftrightarrow \{U_\alpha, s_\alpha : U_\alpha \longrightarrow \mathbb{C}^r, s_\alpha(x) = g_{\alpha\beta}(x)s_\beta(x)\} \tag{40}$$

Now, we discuss some examples of holomorphic vector bundles:

Example 9 (Tangent Bundle)**.** The *tangent bundle* is defined as

$$TX := \coprod_{a\in X} T_a X, \quad v_a \overset{\pi}{\longmapsto} a \tag{41}$$

and the transition functions are given by $g_{\alpha\beta} = (J^{-1}(z_\alpha \circ z_\beta^{-1}))^t$, whereas the sections are vector fields: $X(a) = \sum_i X^i(a)\partial_{x_i}\big|_a \in \Gamma(U, TX)$

Example 10 (Cotangent Bundle)**.** The *cotangent bundle* is defined as

$$\Omega^1_X := \hom\left(TX, X \times \mathbb{C}\right) = T^*X \tag{42}$$

and is dual to the tangent bundle. Its transition functions are given by $g_{\alpha\beta} = J(z_\alpha \circ z_\beta^{-1})$ and the sections are holomorphic 1-forms: $\omega(a) = \sum_i \omega_i(x)dx^i\big|_a$

Remark. We have

$$\left(E \longleftrightarrow (U_\alpha, g_{\alpha\beta})\right) \Longleftrightarrow \left(E^* \longleftrightarrow (U_\alpha, {g^t_{\alpha\beta}}^{-1}\right). \tag{43}$$

Also, we can take direct sums $E \oplus F$, tensor products $E \otimes F$, exterior products, etc. to construct new vector bundles.

This leads to a last example:

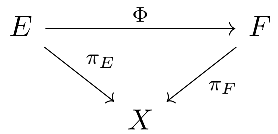
Example 11 (Canonical Bundle)**.** The *canonical bundle* is defined via the determinant:

$$K_X := \det\left(\Omega^1_X\right) = \bigwedge^{\dim(X)} \Omega^1_X \tag{44}$$

It is a *line bundle*, that is a vector bundle of rank one.

Definition 23 (Morphism of Vector Bundles)**.** For vector bundles $\pi_E : E \to X$ of rank r and $\pi_F : F \to X$ of rank k, a map $\Phi : E \to F$ is called a morphism of vector bundles if

1) it commutes with the projections: $\pi_F \circ \Phi = \pi_E$,



2) it is linear on the fibers: $\Phi_X : E_X \to F_X$ is linear,

3) it has constant rank: $\operatorname{rank}\Phi_X$ does not depend on $x \in X$.

Local Representation of Morphisms: It follows that we have a map $(x, v) \mapsto (x, \Phi_\alpha(x)v)$ with $\Phi_\alpha : U_\alpha \to \mathrm{Mat}(l \times r, \mathbb{C})$, giving the local representation. We have

$$\begin{array}{ccc} E\big|_{U_\alpha} & \xrightarrow{\Phi|_{U_\alpha}} & F\big|_{U_\alpha} \\ \big\downarrow \psi_\alpha & & \big\downarrow \psi_\alpha \\ U_\alpha \times \mathbb{C}^r & \xrightarrow{\Phi_\alpha} & U_\alpha \times \mathbb{C}^l \end{array},$$

where Φ_α acts as the identity on the first component and is linear on the second component.

Change of Trivialisation: Just like before, one has

$$\begin{cases} \Phi(x) = \psi_\alpha^{-1}\left(x, \Phi_\alpha(x)v\right) \\ \varphi_\alpha\left(\Phi(x)\right) = \left(x, \Phi_\alpha(x)v\right) \end{cases}$$

and

$$\begin{aligned} (x, \Phi_\alpha(x)v) &= \varphi_\alpha \circ \Phi \circ \psi_\alpha^{-1}(x, v) = \varphi_\alpha \circ \left(\varphi_\beta^{-1} \circ \varphi_\beta\right) \circ \Phi \circ \left(\psi_\beta^{-1} \circ \psi_\beta\right) \circ \psi_\alpha^{-1}(x, v) \\ &= \varphi_\alpha \circ \varphi_\beta^{-1} \circ \left(\varphi_\beta \circ \Phi \circ \psi_\beta^{-1}\right) \circ \psi_\beta \circ \psi_\alpha^{-1}(x, v) \\ &= \varphi_\alpha \circ \varphi_\beta^{-1} \circ (x, \Phi_\beta(x) \circ g_{\beta\alpha}(x)v) \\ &= (x, h_{\alpha\beta}(x) \circ \Phi_\beta(x) \circ g_{\beta\alpha}(x)v)\,, \end{aligned}$$

so

$$\Phi : E \to F \longleftrightarrow \left\{\Phi_\alpha : U_\alpha \to \mathrm{Mat}(r \times l, \mathbb{C}) : \Phi_\alpha(x) = \underbrace{h_{\alpha\beta}}_{F}(x)\Phi_\beta(x)\,\underbrace{g_{\beta\alpha}}_{E}(x)\right\}.$$

Note that this is just the "change of basis" of a matrix: $\Phi' = h\Phi g^{-1}$

Remark (Injective Morphisms)**.** The map $\Phi : E \hookrightarrow F$ (with $\operatorname{rank} E = r \leq l = \operatorname{rank} F$) is injective if it behaves like an inclusion, i.e. there exist trivialisations such that

$$\begin{aligned} \varphi_\alpha \circ \Phi \circ \psi_\alpha^{-1} : U_\alpha \times \mathbb{C}^r &\longrightarrow U_\alpha \times \mathbb{C}^l \\ \big(x, (v_1, \ldots, v_r)\big) &\longmapsto \big(x, (v_1, \ldots, v_r, \underbrace{0, \ldots, 0}_{l-r})\big). \end{aligned} \tag{45}$$

For Φ injective one can write "nice" transition functions,

$$\left(\begin{array}{c|c} g_{\alpha\beta}(x) & * \\ \hline * & h_{\alpha\beta}(x) \end{array}\right)$$

where $g_{\alpha\beta}$ is the transition function of E and $k_{\alpha\beta}$ is the transition function of F.

The above situation is represented by a short exact sequence

$$0 \longrightarrow E \xrightarrow[\text{injective}]{i} F \xrightarrow[\text{surjective}]{\pi} F/E \longrightarrow 0 \tag{46}$$

with $\mathrm{Im}(i) = \ker(\pi)$.

Definition 24 (Pull-back Bundle)**.** Let $f : Y \to X$ be holomorphic and $E \leftrightarrow (U_\alpha, g_{\alpha\beta})$ be a vector bundle on X. Then f induces a fiber bundle on Y by composition, given by $f^*E \leftrightarrow (f^{-1}(U_\alpha), g_{\alpha\beta} \circ f)$. This is the *pull-back bundle*. Note that $E_{f(x)} = f^*E_x$.
Regarding submanifolds, for the inclusion $i : Y \hookrightarrow X$, we write $E\big|_Y := i^*E$ and note that $E\big|_Y \leftrightarrow (Y \cap U_\alpha, g_{\alpha\beta}\big|_{U_\alpha \cap U_\beta \cap Y})$.

Definition 25 (Normal Bundle)**.** For an inclusion $i : Y \hookrightarrow X$, consider $TX\big|_Y := i^*TX$. Then the *normal bundle* is given by $\mathcal{N}_{Y/X} := TX\big|_Y / TY$. Alternatively, one can look at an short exact sequence:

$$0 \longrightarrow TY \xrightarrow{d_i} TX\big|_Y \longrightarrow TX\big|_Y \Big/ TY \longrightarrow 0 \tag{47}$$

Theorem 19 (Adjunction Formula)**.** *Let $Y \hookrightarrow X$ be a complex submanifold. Then*

$$K_Y \cong K_X\big|_Y \otimes_{\mathcal{O}_X} \det \mathcal{N}_{Y/X}. \tag{48}$$

Proof. Just take the determinant of the normal bundle sequence:

$$\begin{gathered} 0 \longrightarrow TY \longrightarrow TX\big|_Y \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0 \\ \implies \det\left(TX\big|_Y\right) \cong \det(TY) \otimes \det\left(\mathcal{N}_{Y/X}\right) \end{gathered}$$

Note that $\det(TX\big|_Y) = \det(TX)\big|_Y$ as $\det(g_{\alpha\beta}\big|_Y) = \det(g_{\alpha\beta})\big|_Y$. Taking the dual yields $K_X\big|_Y \cong K_Y \otimes \det(\mathcal{N}_{Y/X})^*$. It follows that $K_Y \cong K_X\big|_Y \otimes \det(\mathcal{N}_{Y/X})$. □

Analogously, consider the dual of the normal bundle exact sequence:

$$0 \longrightarrow \mathcal{N}^*_{Y/X} \longrightarrow T^*X\big|_Y \longrightarrow T^*Y \longrightarrow 0$$

This is the *canonical exact sequence.*

Remark. Later, we will see a special case of this for codimension one hypersurfaces in $\mathbb{P}^n$.

We will now see the relation between vector bundles and sheaves:

4.2 The Relation of Holomorphic Vector Bundles and (locally free) Sheaves

Definition 26 (Sheaf of Sections of "E)**.** Let $\pi : E \to X$ be a holomorphic vector bundle. We define the *sheaf of sections* of E:

$$U \longmapsto \mathcal{E}(U) := \left\{ s : U \longrightarrow \pi^{-1}(E) : \pi \circ s = \mathrm{id},\ s \text{ is holomorphic} \right\} \tag{49}$$

It is a sheaf of $\mathcal{O}_X$-modules (here, $\mathcal{O}_X$ is the sheaf of sections of the trivial bundle $X \times \mathbb{C}$).

Theorem 20. *There exists a bijection between holomorphic vector bundles of rank r and locally free sheaves of rank r.*

"Proof". Remember that $\mathcal{E}$ is locally free of rank r if $\mathcal{E}\big|_U \cong \mathcal{O}_X^{\oplus r}\big|_U$. Clearly, $\mathcal{E}$ is locally free as E is locally isomorphic to $U \times \mathbb{C}^r$. Also, by choosing the trivialisation $\psi_i : \mathcal{E}\big|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r}$, the transition maps $\psi_{ij} := \psi_i \circ \psi_j^{-1} : \mathcal{O}_{U_i \cap U_j}^{\oplus r} \xrightarrow{\cong} \mathcal{O}_{U_j \cap U_i}^{\oplus r}$ are given by a multiplication with a matrix of holomorphic functions on $U_i \cap U_j$. This constructs $U \leftrightarrow (U_i, \psi_{ij})$. □

5 Cohomology

Actually Čech cohomology.

5.1 Čech Cohomology

Definition 27 (p-th Cochain)**.** Let X be a topological space with an open covering $U = \{U_i\}_{i\in I}$ such that $X = \bigcup_{i\in I} U_i$. For $q = 0, 1, \dots$ and a sheaf $\mathcal{F}$ we define the *q-th cochain group* of $\mathcal{F}$:

$$\mathcal{C}^q\left(U, \mathcal{F}\right) := \prod_{\substack{(i_0,\dots,i_q)\in I^{q+1} \\ (i_1<\dots<i_q)}} \mathcal{F}\left(U_{i_0} \cap \dots \cap U_{i_q}\right) \tag{50}$$

The elements of $\mathcal{C}^q(U, \mathcal{F})$ are called *q-cochains*: they are given by a family of section as follows:

$$\left(f_{i_0 \cdots q}\right)_{i_0,\dots,i_q \in I^{q+1}} : f_{i_0,\dots,i_q} \in \mathcal{F}\left(U_{i_0} \cap \dots \cap U_{i_q}\right) \ \forall (i_0, \dots, i_q) \in I^{q+1},\ i_0 < \dots < i_q \tag{51}$$

Note. $\mathcal{C}^q$ is indeed a group with component-wise addition.

Definition 28 (q-Cohomology Operator)**.** We define a cohomology operator on $\mathcal{C}^q(U, \mathcal{F})$:

$$\begin{aligned} \delta : \mathcal{C}^q\left(U, \mathcal{F}\right) &\longrightarrow \mathcal{C}^{q+1}\left(U, \mathcal{F}\right) \\ (f)_{i_0,\dots,i_q} &\longmapsto (\delta f)_{i_0,\dots,i_{q+1}} = \sum_{k=0}^{q+1} (-1)^k f_{i_0 \dots \hat{i_k} \cdots q+1}\Big|_{U_{i_0} \cap \dots \cap U_{q+1}} \end{aligned} \tag{52}$$

(Use the restriction morphism of $\mathcal{F}$). Note that $f_{i_0 \dots \hat{f_k} \cdots q+1} \in \mathcal{F}(U_{i_0 \cap \dots \cap \widehat{U_{i_k}} \cap \dots \cap \widehat{U_{q+1}}})$, so we restrict to the intersection $U_{i_0} \cap \dots \cap U_{i_{q+1}}$ as to get an element in $\mathcal{F}(U_{i_0} \cap \dots \cap U_{i_{q+1}})$.
We have $\delta^2 = \delta \circ \delta = 0$, δ is **nilpotent**!

Example 12. Consider $\mathcal{C}^0(U, \mathcal{F})$, $\mathcal{C}^1(U, \mathcal{F})$ and δ. Explicitly, one has

$$\begin{aligned}
&\mathcal{C}^0(U, \mathcal{F}) = \mathcal{F}(U_0) \times \mathcal{F}(U_1) \times \dots = \prod_{i\in I} \mathcal{F}(U_i) \ni (f_i)_{i\in I} \\
&\mathcal{C}^1(U, \mathcal{F}) = \mathcal{F}\left(U_0 \cap U_1\right) \times \mathcal{F}\left(U_0 \cap U_2\right) \times \dots = \prod_{i,j\in I} \mathcal{F}\left(U_i \cap U_j\right) \ni (f_{ij})_{i,j\in I} \\
&\delta^0 : \mathcal{C}^0(U, \mathcal{F}) \to \mathcal{C}^1(U, \mathcal{F}) \ : \quad (\delta f)_{ij} = f_j\big|_{U_i \cap U_j} - f_i\big|_{U_i \cap U_j} \equiv g_i j \\
&\delta^1 : \mathcal{C}^1(U, \mathcal{F}) \to \mathcal{C}^2(U, \mathcal{F}) \ : \quad (\delta f)_{ijk} = f_{jk}\big|_{U_i \cap U_j \cap U_k} - f_{ik}\big|_{U_i \cap U_j \cap U_k} + f_{ij}\big|_{U_i \cap U_j \cap U_k} \equiv g_{ijk}
\end{aligned}$$

Note that one can easily verify the nilpotency in this case

$$(f_i) \overset{\delta^0}{\longmapsto} f_j - f_i \overset{\delta^1}{\longmapsto} (f_j - f_i) - (f_k - f_i) + (f_k - f_j) = 0,$$

so indeed $\delta^1 \circ \delta^0 = 0$.

Definition 29 (q-Cocycles/q-Coboundaries)**.** Since δ is a group homomorphism, we define

- *q-cocycles*: $Z^1(U, \mathcal{F}) := \ker(\delta : \mathcal{C}^q \to \mathcal{C}^{\text{II}+\infty})$
- *q-coboundaries*: $B^q(U, \mathcal{F}) := \text{Im}(\delta : \mathcal{C}^{q-1} \to \mathcal{C}^q)$

Note that since $\delta^2 = 0$, we have $\alpha \in B^q \implies \alpha \in Z^{q+1}$.

Example 13. Consider $Z^0(U, \mathcal{F})$ and $Z^1(U, \mathcal{F})$. By the very definition one has

(i) $(f_i) \in Z^0 \Leftrightarrow (\delta f)_{ij} = 0 \; \forall i, j \Leftrightarrow f_j\big|_{U_i \cap U_j} = f_i\big|_{U_i \cap U_j}$, so there exists an $f \in \mathcal{F}(X)$, a global section (compare with axiom 2) for sheaves).

(ii) $(f_{ij}) \in Z^1 \Leftrightarrow (\delta f)_{ijk} = 0 \forall i, j, k \Leftrightarrow \underbrace{f_{ik}\big|_{U_i \cap U_j \cap U_k} = f_{ij}\big|_{U_i \cap U_j \cap U_k} + f_{jk}\big|_{U_i \cap U_j \cap U_k}}_{\text{cocycle relations}}$

It follows that $f_{ii} = 0$ (using $i = j = k$) and $f_{ij} = -f_{ji}$ (using $i = k$).

Note that one can take the quotient as usual. This leads to:

Definition 30 (q-(Čech) cohomology group)**.** The *q-cohomology group* of $\mathcal{F}$ with respect to the covering U is given by

$$H^q(U, \mathcal{F}) := Z^q(U, \mathcal{F}) \big/ B^q(U, \mathcal{F}). \tag{53}$$

(Analogously $\check{H}^q(U, \mathcal{F}) := h^q(\mathcal{C}^0(U, \mathcal{F}))$.

Remark (A little philosophy)**.** This cohomology theory is very suitable for computations and it does not require "acyclic" sheaves to be defined. The problem is that it depends on the covering: like in the definition of the stalk of a sheaf, one should take finer and finer coverings and pass to the limit $\check{H}^q(X, \mathcal{F}) := \lim_{\to} \check{H}^q(U, \mathcal{F})$. This cohomology theory coincides with the "true" sheaf cohomology if X is a "descent" topological space (e.g. it is paracompact). In this case $H^q(X, \mathcal{F}) \cong \check{H}^q(X, \mathcal{F}) = \lim_{\to} \check{H}^q(U, \mathcal{F})$.

The Meaning of Cohomology: We now consider cohomology groups in some details:

1. $H^0(X, \mathcal{F}) = Z^0(X, \mathcal{F})$: global sections of $\mathcal{F}$, i.e. $H^0(X, \mathcal{F}) = \mathcal{F}(X)$. Note that $H^0(X, \mathcal{O}_X) \cong \mathbb{C}$ for X compact and connected. Also notice that it is independent of the covering.
2. $H^{i>0}(X, \mathcal{F})$: in order to see the meaning of the higher cohomology groups one should introduce (short) exact sequences of sheaves! We first recall the following facts:

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then:

(i) $\ker(\varphi) := \{U \mapsto \ker(\varphi)(U) := \ker(\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U))\}$, a sheaf!

(ii) $\mathrm{Im}(\varphi) := \{U \mapsto \mathrm{Im}(\varphi)(U) := \mathrm{Im}(\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U))\}$, not a sheaf!

$\mathrm{Im}(\varphi)$ and $\mathrm{coker}(\varphi)$ are only presheaves in general: we consider their (sheafified) "associated" sheaf.

5.2 Exact Sequences of Sheaves

Definition 31 (Exact sequence (of sheaves))**.** Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X and let $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ be the induced morphism on the stalks. Then a sequence of sheaves

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \tag{54}$$

is called *exact* if for each $x \in X$, the sequence $\mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x$ is exact, i.e. if $\mathrm{Im}(\alpha_x) = \ker(\beta_x)$. In particular, we say that $\alpha : \mathcal{F} \to \mathcal{G}$ is *injective* or a *monomorphism* if $0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ is exact ($\ker(\alpha_x) = 0 \; \forall x \in X$). We say that $\alpha : \mathcal{F} \to \mathcal{G}$ is *surjective* or an *epimorphism* if $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \longrightarrow 0$ is exact ($\mathrm{Im}(\alpha_x) = \mathcal{G}_x \; \forall x \in X$).
An exact sequence of the form

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0 \tag{55}$$

is called a *short exact sequence.*

Lemma 3. *Let $\alpha : \mathcal{F} \to \mathcal{G}$ be injective. Then for every $U \subseteq X$ $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective. In particular, then $\alpha(X) : \mathcal{F}(X) = H^0(X, \mathcal{F}) \to \mathcal{G}(X) = H^0(X, \mathcal{G})$ is injective, too.*

Proof. We let $f \in \mathcal{F}(U)$ with $\alpha_U(f) = 0$. We want to show that $f = 0$. Since $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$, every $x \in U$ has a neighbourhood $V_x \subseteq U$ s.t. $f\big|_{V_x} = 0$, but then by (sheaf) axiom 1 (local identity) $f = 0$ in U. Hence, $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective. □

Warning: If $\alpha : \mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ is surjective, it is not necessarily true that $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all $U \subseteq X$!

Example 14. Consider $X = \mathbb{C}^*$ with

$$\exp : \mathcal{O}_X \longrightarrow \mathcal{O}^*_X, \qquad f \longmapsto \exp(2\pi i f). \tag{56}$$

Let $U_1 = C^* \setminus \mathbb{R}_-$ and $U_2 = \mathbb{C}^* \setminus R_+$ and prove that it is surjective. Then the positive axes can be seen as two possible *branch cuts* in $\mathbb{C}^*$ but these cannot be crossed locally in U_1 and U_2 (they are simply connected), so the *complex logarithm* is a single valued well-defined function: We define $U_\alpha \supseteq U_1 \mapsto \log_U \in \mathrm{Hom}\left(\mathcal{O}^*_{\mathbb{C}^*}(U), \mathcal{O}_{\mathbb{C}^*}(U)\right)$ with $f \mapsto \log_U(f) \equiv \frac{1}{2\pi i} \log_U(f)$. In particular, posing

$f_i \equiv \log_{U_i}(g_i)$ for $g_i \in \mathcal{O}^*_{\mathbb{C}^*}(U)$, we have $\exp_{U_i}(f_i) = g_i$. Then exp is locally surjective, i.e. $\forall U \subseteq X$ and $f \in \mathcal{O}^*_{\mathbb{C}^*}(U)$, there exists $x \in U$ and $V_x \subseteq U$ such that $f|_{V_x}$ admits a preimage with respect to $\exp_{V_x} \in \mathrm{Hom}\left(\mathcal{O}(V_x), \mathcal{O}^*(V_x)\right)$: this implies surjectivity at the level of the stalks, indeed if $g_x \in \mathcal{O}^*_{\mathbb{C}^*,x}$ for some $x \in U_i$, then we represent g_x by $g \in \mathcal{O}^*_{\mathbb{C}^*}(V_x)$ with $V_x \subseteq U_i$ but since $\exp_{V_x}$ is surjective then there exists $f \in \mathcal{O}_{\mathbb{C}^*}(V_x)$ such that $g = \exp_{V_x}(f)$. It follows that $g_x = \left(\exp_{V_x}(f)\right)_x = \exp_x(f_x)$ which concludes the verification.

On the other hand consider the function $z \mapsto f(z) = z \in \mathcal{O}^*_{\mathbb{C}^*}(\mathbb{C}^*)$: Then, there is no $f \in \mathcal{O}_{\mathbb{C}^*}(\mathbb{C}^*)$ such that $z = \exp_{\mathbb{C}^*}(f)$ because $\log_{\mathbb{C}^*}(z)$ is not single valued!

Lemma 4. *If $0 \to \mathcal{F} \stackrel{\alpha}{\to} \mathcal{G} \stackrel{\beta}{\to} \mathcal{H}$ is exact, then $0 \to \mathcal{F}(U) \stackrel{\alpha_U}{\to} \mathcal{G}(U) \stackrel{\beta_U}{\to} \mathcal{H}(U)$ is exact for all $U \in X$.*

Proof. We have already proved that $0 \to \mathcal{F}(U) \to \mathcal{G}(U)$ is exact. We need to prove that $\mathrm{Im}(\alpha_U) = \ker(\beta_U)$.

1. $\mathrm{Im}(\alpha_U) \subseteq \ker(\beta_U)$: Let $f \in \mathcal{F}(U)$ and let $g = \alpha_U(f) \in \mathrm{Im}(\alpha_U)$. Since the sequence $0 \to \mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$ is exact for all $x \in X$, then each point x has a neighborhood $V_x \subseteq U$ such that $\beta_U(g)\big|_{V_x} = 0$ by exactness. Then, by sheaf axiom (I) one has that $\beta_U(g) = 0$ and hence $g \in \ker(\beta_U)$.

2. $\mathrm{Im}(\alpha_U) \supseteq \ker(\beta_U)$: Suppose $g \in \mathcal{G}(U)$ such that $\beta_U(g) = 0$, i.e. $g \in \ker(\beta_U)$. Since for all $x \in X$ $\ker(\beta_x) = \mathrm{Im}(\alpha_x)$, then there is an open cover $U = \bigcup_l V_l$ and elements $f_l \in \mathcal{F}(V_l)$ such that $\alpha_{V_l}(f_l) = g\big|_{V_l}$. Then, in $V_l \cap V_j$ one has $\alpha_{V_l \cap V_j}(f_l - f_j) = g\big|_{V_l \cap V_j} - g\big|_{V_l \cap V_j} = 0$, hence since α is injective $f_l = f_j$ for all i, j on $V_l \cap V_j$. Then it follows from sheaf axiom (II) that there exists $f \in \mathcal{F}(U)$ with $f\big|_{V_i} = f_i$ $\forall i$. Then, since $\alpha_U(f)\big|_{V_i} = \alpha_U(f|_{V_i}) = g|_{V_i}$, sheaf axiom (I) implies that $\alpha(f) = g$.

□

Remark (Global Sections Functor)**.** Given a (complex) manifold one can define a functor as follows:

$$\begin{aligned} (\cdot)(X) : \mathrm{Sh}_X &\longrightarrow \mathrm{Ab}, \\ \mathcal{F} &\longmapsto \mathcal{F}(X) \end{aligned} \tag{57}$$

This functor is left exact/preserves injectivities but it is not right exact: it does not preserve surjectivitites:

$$[0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0] \longmapsto [0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to ?]$$

In general sheaf cohomology quantifies the failure for this functor to be exact: Čech cohomology is a way to compute sheaf cohomology. The problems with surjectivity come from examples such as those of $\exp : \mathcal{O} \to \mathcal{O}^*$.

We now study this in the framework of Čech cohomology.

Remark (Induced Morphisms in Cohomology). Let us consider $\alpha : \mathcal{F} \to \mathcal{G}$. Then we have corresponding morphisms in cohomology: $\alpha^q : H^q(X, \mathcal{F}) \to H^q(X, \mathcal{G})$.

$q = 0$: One simply has $\alpha^0 : \mathcal{F}(X) = H^0(X, \mathcal{F}) \to G(X) = H^0(X, \mathcal{G})$.

$q = 1$: Let $\{U_i\} = 0$ be a covering $\bigcup_i U_i = X$. We consider $\alpha_U : \mathcal{C}^1(U, \mathcal{F}) \to \mathcal{C}^1(U, \mathcal{G})$ such that $(f_{ij}) \mapsto \alpha_U(f_{ij}) := \big(\alpha_{U_i \cap U_j}(f_{ij})\big)_{ij} \in \mathcal{C}^1(U, \mathcal{G})$. The map takes cocycles in cocycles and coboundaries in coboundaries, hence it descends in cohomology: $\alpha_U \mapsto [\alpha_U] : H^1(U, \mathcal{F}) \to H^1(U, \mathcal{G})$. As usual, taking the limit over U one gets $\alpha^1 : H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G})$.

$q > 1$: Exactly the same way!

<u>Construction:</u> "Connecting Homomorphism": Suppose we have

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0. \tag{58}$$

Then we can construct a map $\delta^0 : H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F})$ as follows:

1. $h \in H^0(X, H)$: Since $\beta_x : \mathcal{G}_x \to \mathcal{H}_x$ is surjective there exists a covering $U = \{U_i\}$ such that $X = \bigcup_i U_i$ and $(g_i) \in \mathcal{C}^0(U, \mathcal{G})$ with $\beta(g_i) = h|_{U_i}$ for all i.
2. Then $\beta(g_j - g_i\big|_{U_i \cap U_j}) = h|_{U_i \cap U_j} - h_{U_i \cap U_j} = 0$ (so $\beta(\delta g) = 0$) which implies $g_j - g_i\big|_{U_i \cap U_j} \in \ker \beta$.
3. By exactness $\ker_U \beta = \operatorname{Im}_U \alpha$ and the previous lemma one has that there exists $f_{ij} \in \mathcal{F}(U_i \cap U_j)$ such that $\alpha_{U_i \cap U_j}(f_{ij}) = g_j - g_i|_{U_i \cap U_j}$.
4. On $U_i \cap U_j \cap U_k$ one has $\alpha_{U_i \cap U_j \cap U_k}(f_{ij} - f_{ik} + f_{jk}) = g_j - g_i - g_k + g_i + g_k - g_j = 0$. Then, by injectivity of α we have $f_{ij} - f_{ik} + f_{jk}|_{U_i \cap U_j \cap U_k} = 0$ and hence $(f_{ij})_{i,j \in I} \in Z^1(U, \mathcal{F})$.
5. We can then define $h \mapsto \delta h \in H^1(X, \mathcal{F})$ where δh is represented by (f_{ij}) constructed as above.

All higher $\delta^{i>0} : H^i(X, \mathcal{H}) \to H^{i+1}(X, \mathcal{F})$ can be constructed analogously!

$$\begin{array}{ccccc}
\mathcal{C}^0(U, \mathcal{G}) \ni & & (g_i) & \xmapsto{\quad\beta\quad} & h \in H^1(U, \mathcal{H}) \\
& & \big\downarrow \delta & & \\
H^1(U, \mathcal{F}) \ni (f_{ij}) & \xmapsto{\alpha} & \alpha(f_{ij}) & \in \mathcal{C}^1(U, \mathcal{G}) &
\end{array}$$

Figure 1: Summary of the maps defining $h \mapsto \delta^0 h$. Note that α is surjective and β is injective.

The connecting homomorphism enters in the following fundamental result:

Theorem 21 (Snake Lemma). *A short exact sequence of sheaves*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0 \tag{59}$$

implies a long exact sequence in cohomology via the connecting homomorphism:

$$
\begin{array}{l}
0 \longrightarrow H^0(X,\mathcal{F}) \xrightarrow{\alpha_0} H^0(X,\mathcal{G}) \xrightarrow{\beta_0} H^0(X,\mathcal{H}) \xrightarrow{\delta^0} \\
\hookrightarrow H^1(X,\mathcal{F}) \xrightarrow{\alpha_1} H^1(X,\mathcal{G}) \xrightarrow{\beta_1} H^1(X,\mathcal{H}) \xrightarrow{\delta^1} \\
\hookrightarrow H^1(X,\mathcal{F}) \xrightarrow{\alpha_2} H^1(X,\mathcal{G}) \xrightarrow{\beta_2} H^1(X,\mathcal{H}) \ \dots
\end{array}
$$

Proof. We only prove the exactness at $H^0(X,\mathcal{H})$:

- $\operatorname{Im}\beta_0 \subseteq \ker\delta^0$: Suppose $g \in H^0(X,\mathcal{G})$ and $h := \beta_0(g)$. In the construction of $\delta^0 h$ we can use $g_i = g|_{U_i}$. But then, since g is a global section, $\delta^0 g = 0$ which implies $\alpha(f_{ij}) = \delta g = 0$ by construction. But since α is injective, we have $f_{ij} = 0$. It follows that $\delta^0(\beta_0(g)) = [f_{ij}] = 0$.

  This is the picture:
  $$
  \begin{array}{ccc}
   & & g & \longmapsto & \beta(g) \\
   & & \downarrow{\scriptstyle\delta} & & \\
  0 & \overset{\alpha}{\rightarrowtail} & \delta g & &
  \end{array}
  $$

- $\ker\delta^0 \subseteq \operatorname{Im}\beta_0$: Suppose δ^0. Then $\delta^0[h_i] = [f_i j] = 0$ and therefore $f_{ij} \in \beta_1(U,\mathcal{F})$ and hence $f_{ij} = \delta^0 f_i = f_j - f_i$. Let us consider $\beta(g_i) = h|_{U_i}$ in the construction of δ^0 with $\delta^0 g_i = \alpha(f_{ij})$. Then $\delta^0(g_i - \alpha(f_i)) = 0$. Indeed:

  $$\delta(g_i - \alpha(f_i)) = \alpha(f_{ij}) - \delta_0\alpha(f_i) = \alpha(f_{ij}) - \big(\alpha(f_j) - \alpha(f_i)\big) = \alpha(f_{ij}) - \alpha(f_{ij}) = 0$$

  and thus $g_i\alpha(f_i) \in Z^0(U,\mathcal{G})$. Also: $\beta(g_i - \alpha(f_i)) = \beta(g_i) - \beta(\alpha(f_i))) = \beta(g_i) = h|_{U_i}$ which finally implies $h \in \operatorname{Im}(\beta_0)$. Diagrammatically:
  $$
  \begin{array}{ccccc}
  f_1 & & g & \overset{\beta}{\longmapsto} & h \\
  \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
  f & \overset{\alpha}{\longmapsto} & \delta g & &
  \end{array}
  $$

□

6 First Applications of Cohomology

In this section, we study some examples.

6.1 Exponential Exact Sequence

$$0 \longrightarrow \mathbb{Z}_X \overset{i}{\hookrightarrow} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1$$

Let us consider the long exact cohomology sequence:

$$0 \longrightarrow H^0(X,\mathbb{Z}) \longrightarrow H^0(X,\mathcal{O}_X) \longrightarrow H^0(X,\mathcal{O}_X^*) \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow \dots$$

Note:

1. $\check{H}^i(X, \mathbb{Z}_X) \cong H^i_{\text{sing}}(X, \mathbb{Z})$. (This is true also more in general...). This means that the part "$\mathbb{Z}_X$" takes care about the topology of X!

2. If X is compact, then $H^1(\mathbb{Z}) \to H^1(\mathcal{O}_X)$ is injective, so that one has two exact sequences: First:

$$0 \to H^0(\mathbb{Z}_X) \to H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_X^*) \to 0$$

If X is also connected, then $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \cong \mathbb{C}/\mathbb{Z} \to 0$. Second:

$$0 \to H^1(\mathbb{Z}_X) \to \underbrace{H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X^*) \cong \text{Pic}(X) \xrightarrow{\delta} H^2(\mathbb{Z}_X)}_{\text{interesting part}} \to \dots$$

Definition 32 (First Chern Class)**.** The *first Chern class* of a holomorphic line bundle $\mathcal{L} \in \text{Pic}(X)$ is the image in $H^2(\mathbb{Z}_X)$ of $\mathcal{L}$ via the boundary map, i.e. $\mathcal{C}_1(\mathcal{L}) := \delta^1\big([g_{ij}]\big) \subseteq H^2(\mathbb{Z}_X)$ where $[g_{ij}] \in H^1(\mathcal{O}_X) \cong \text{Pic}(X)$.

This is the most important characteristic class of a holomorphic line bundle.

Example 15 ($\mathbb{P}^n$ and Exponential Exact Sequence)**.** Remember that

$$H^i(\mathbb{P}^n, \mathbb{Z}_{\mathbb{P}^n}) \cong \begin{cases} \mathbb{Z} & i = 2n \\ 0 & \text{else} \end{cases}$$

. Now

1. $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 0$,

2. $0 \to H^1(\mathbb{P}^n, \mathbb{Z}) \cong 0 \to H^1(\mathbb{P}^n, \mathcal{O}_X) \cong 0 \to \text{Pic}(\mathbb{P}^n) \xrightarrow{\delta^1} H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z} \to 0$.

It follows that

$$\begin{aligned} \deg : \text{Pic}(\mathbb{P}^n) &\xrightarrow{\cong} \mathbb{Z} \\ [\mathcal{O}^n_{\mathbb{P}}(k)] &\longmapsto k \end{aligned} \tag{60}$$

Remark (Cohomology of $\mathcal{O}_{\mathbb{P}^n}(k)$)**.** The previous result suggests that one can study the cohomology of the line bundles $\mathcal{O}_{\mathbb{P}^n}(k)$ for any $n > 0$, for all $k \in \mathbb{Z}$. This is achieved by Čech cohomology computations using the standard covering of $\mathbb{P}^n$. Let us see a couple of examples over $\mathbb{P}^1$:

1. $\underline{H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))}$: Recall that $U_0 = \{[x_0 : x_1] \mid X_0 \neq 0\}$ and we set $z := \frac{x_1}{x_0}$ the corresponding local coordinate on U_0. A generic section of $\mathcal{O}_{\mathbb{P}^1}(2)\big|_{U_0}$ will be of the form $s_0 = f(z)\, e_{U_0}$ where $f : U_0 \to \mathbb{C}$ is a holomorphic function and e_{U_0} is a local basis of $\mathcal{O}_{\mathbb{P}^1}(2)$. Similarly, a generic

section of $\mathcal{O}_{\mathbb{P}^1}(2)\big|_{U_1}$ will be $s_1 = g(w)\; e_{U_1}$ (with $w = \frac{1}{z}$). In the intersection $U_0 \cap U_1 = \{[x_0 : x_1] \mid x_0 \neq 0 \neq x_1\}$ one has $e_{U_1} = z^2 e_{U_0}$ so that if $s_i \equiv (s_0, s_1)$ is a 0-cochain $\mathcal{C}^0(\{U_0, U_1\}, \mathcal{O}_{\mathbb{P}^1}(2))$

$$
\begin{aligned}
0 \overset{!}{=} (\delta s)_{01} = s_1 - s_0\big|_{U_0 \cap U_1} &= g(w)\; e_{U_1} - f(z)\; e_{U_0} = g(w)\; e_{U_1} - f\left(\frac{1}{w}\right)\; w^2 e_{U_1} \\
&= \left(\sum_{l=0}^{\infty} g_l w^l - \sum_{j=0}^{\infty} f_j w^{-j+2}\right) e_{U_1} \\
&= \left((g_0 - f_2) + (g_1 - f_1)w + (g_2 - f_0)w^2\right) + \sum_{l>2} g_l w^l + \sum_{j>2} f_j w^{-l}
\end{aligned}
$$

so every coefficient has to vanish separately:

$$
\begin{aligned}
s \in H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)\big) \quad &\Longleftrightarrow \quad s = \left(a + bz + cz^2, c + bw + aw^2\right) \\
&\Longleftrightarrow s = \left(x_0^2\left(a + b\frac{x_1}{x_0} + c\left(\frac{x_1}{x_0}\right)^2\right), x_1^2\left(c + b\frac{x_0}{x_1} + a\left(\frac{x_0}{x_1}\right)^2\right)\right) \\
&\Longleftrightarrow s = ax_0^2 + bx_0 x_1 + cx_1^2
\end{aligned}
$$

In other words $s \in H^0(\mathcal{O}_{\mathbb{P}^1}(2))$ is a *homogeneous polynomial of degree 2*!

2. $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$: Left as an exercise. One should find

$$
H^1\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)\right) = \left\langle \frac{1}{x_0 x_1} \right\rangle_{\mathbb{C}}.
$$

In general, one can compute the dimensions of the cohomology groups for $\mathbb{P}^n$:

$$
h^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)\right) := \dim H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)\right) = \binom{k+n}{n} \tag{61}
$$

for $k \geq 0$ and

$$
h^n\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})\right) := \dim H^n\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)\right) = \binom{-k-1}{-k-n-1} \tag{62}
$$

for $k \geq -n-1$.

6.2 Euler Exact Sequence

$$
0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(+1)^{\oplus n+1} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0
$$

Let us again consider the long exact cohomology sequence:

$$
0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n}) \cong \mathbb{C} \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(+1))^{\oplus n+1} \cong (\mathbb{C}^{n+1})^{\oplus n+1} \longrightarrow H^0(T_{\mathbb{P}^n}) \longrightarrow H^1(\mathcal{O}_{\mathbb{P}^n}) \equiv 0
$$

That means we have

$$
0 \to \mathbb{C} \to \mathbb{C}^{(n+1)^2} \to H^0(T_{\mathbb{P}^n}) \to 0 \implies H^0(T_{\mathbb{P}^n}) \cong \mathbb{C}^{(n+1)^2-1}
$$

Meaning: "Infinitesimal Automorphisms": $H^0(T_X)$ parameterises the infinitesimal automorphisms of X, in particular in the case of $\mathbb{P}^n$ we have $\mathrm{Aut}(\mathbb{P}^n) = PSL(n, \mathbb{C})$ so that $H^0(X, T_{\mathbb{P}^n}) \cong \mathfrak{pgl}(n, \mathbb{C})$ (where $\mathfrak{pgl}(n, \mathbb{C})$ is the Lie algebra.)

Going up in the long exact sequence we find $H^{i>1}(T_{\mathbb{P}^n}) = 0$. The remarkable case is given by H^1. Meaning: "infinitesimal Deformations:" $H^1(T_X)$ parameterises the infinitesimal deformations of X. In particular, in the case of $\mathbb{P}^n$, we find no deformations. In this case we say that the complex manifold is *rigid*.

Remark. One can understand the maps entering in the Euler exact sequence as follows:

1. $\mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(+1)^{\oplus n+1}$ with $f \longmapsto (x_0 f, x_1 f, \ldots, x_n f)$

2. $\mathcal{O}_{\mathbb{P}^n}^{\oplus n+1}(+1) \longrightarrow T_{\mathbb{P}^n}$ with $(s_0, \ldots, s_n) \longmapsto \sum_{k=0}^n s_k \partial_{x_k}$

Exercise: Why is this exact?

6.3 Normal Exact Sequence and Adjunction

Recall the definitions of the pull-back bundle 24, the normal bundle 25 and the canonical bundle 11.

Definition 33 (First Chern Class of a Complex Manifold)**.** Let X be a complex manifold. Then we define the *(first) Chern class* of X to be $\mathcal{C}_1(X) := \mathcal{C}_1(K_X)$ where K_X is the canonical bundle of X. Also $\mathcal{C}_1(\bigwedge^n TX)$.

Also recall the adjunction formula 19.

Note. 1. We want to study this for dimension 1 hypersurfaces Y in X, this means $\dim Y = \dim X - 1$.

2. In particular, we want to study codimension 1 hypersurfaces in $\mathbb{P}^n$, these hypersurfaces are called *divisors*.

Fact: Hypersurfaces of codimension 1 are always given by the zero locus of a holomorphic global section of some line bundle ("divisor-line bundle correspondence").

We recall the following facts for codimension 1 hypersurfaces:

1. If $\dim Y = \dim X - 1$, then if $a \in Y$ there exists $(U, z = (z_1, \ldots, z_n))$ such that $Y \cap U = \{x \in U \mid z_n(x) = 0\}$.

2. A local equation for Y is a pair (U, f) with $f : U \to \mathbb{C}$ holomorphic such that

   - $Y \cap U = \{x \in U \mid f(x) = 0\}$,

- if $g \in \mathcal{O}(U)$ and $g(U \cap Y) = 0 \implies g = hf$ with $h \in \mathcal{O}_X^*(U)$.

Lemma 5. *(U, z_n) is a local equation for Y.*

3. If (U_α, f_α) and (U_β, f_β) are two local equations for $Y \hookrightarrow X$, then $f_\alpha/f_\beta \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. This allows to introduce a line bundle: $\mathcal{L}_Y \xrightarrow{\pi} X$ such that given an open covering $\{U_\alpha\}_{\alpha \in I}$ of X, one has $\mathcal{L}_Y \leftrightarrow (U_\alpha, f_\alpha/f_\beta)$.

   Remark. $\mathcal{L}_Y$ does not depend on the choice of local equations for Y: indeed if one has $\tilde{\mathcal{L}}_Y \leftrightarrow (U_\alpha, h_\alpha/h_\beta)$ for local equations $h_\alpha = 0$, then $\Phi_\alpha := h_\alpha/f_\alpha : U_\alpha \to \mathbb{C}^*$ (same class in $H^1(\mathcal{O}^*)$) and $g_{\alpha\beta} := f_\alpha/f_\beta = \Phi_\alpha(h_\alpha/h_\beta)\Phi_\beta^{-1} = \Phi_\alpha \tilde{g}_{\alpha\beta}\Phi_\beta^{-1}$.

4.

   Theorem 22. *Let $Y \hookrightarrow X$ be a hypersurface and let $\mathcal{L}_Y$ as above. Then:*

   - *There exists $s \in H^0(X, \mathcal{L}_Y)$ such that $Y = \{x \in X \mid s(x) = 0\}$ (zero locus).*
   - *There exists a covering $\{U_\alpha\}$ of X with $s \leftrightarrow \{s_\alpha : U_\alpha \to \mathbb{C}\}$ such that (U_α, s_α) is a local equation for Y.*
   - *If $\mathcal{L}$ is a line bundle with $s \in H^0(X, \mathcal{L})$ which gives a family of local equations for Y, then $\mathcal{L} \cong \mathcal{L}_Y$.*

Notation: $\mathcal{L}_Y \cong \mathcal{O}_X(D)$ in the context of the divisors/line bundle correspondence.

Theorem 23. *Let $Y \overset{i}{\hookrightarrow} X$ and let $\mathcal{L}_Y$ as above. Then we have $\mathcal{N}_{Y/X} \cong \mathcal{L}_Y\big|_Y$.*

Proof. We note that in this case $\mathcal{N}_{Y/X}$ is a line bundle since $\mathrm{codim}(Y) = 1$.

- From adjunction we have $K_Y \cong K_X\big|_Y \otimes \mathcal{N}_{Y/X}$ so that dualising $\det(TY) \cong \det(TX\big|_Y) \otimes \mathcal{N}^*_{Y/X}$ gives $\mathcal{N}_{Y/X} \cong \det(TX\big|_Y) \otimes \det(TX)^*$.
- Let us now choose local charts $(U_\alpha, z_\alpha := z_{\alpha,1}, \dots, z_{\alpha,n})$ such that $(U_\alpha, z_{\alpha,n})$ is a local equation for Y. It follows that an atlas for Y is given by $\mathcal{A}_Y = (Y \cap U_\alpha, z_{\alpha,1}, \dots, z_{\alpha,n-1})_{\alpha \in I}$.
- The fiber bundles that appear are given as follows:

$$
\begin{aligned}
TY &\leftrightarrow \left\{U_\alpha \cap Y,\ g_{\alpha\beta} = \frac{\partial z_{\alpha,k}}{\partial z_{\beta,l}},\ k,l = 1,\dots,n-1\right\} \rightsquigarrow \mathrm{rank}(TY) = n-1,\\
TX\big|_Y &\leftrightarrow \left\{U_\alpha \cap Y,\ G_{\alpha\beta} = \frac{\partial z_{\alpha,k}}{\partial z_{\beta,l}},\ k,l = 1,\dots,n\right\} \rightsquigarrow \mathrm{rank}\left(TY\big|_Y\right) = n,\\
\mathcal{L}_Y &\leftrightarrow \left\{U_\alpha, h_{\alpha\beta} = \frac{\partial z_{\alpha,n}}{\partial z_{\beta,n}}\right\} \rightsquigarrow \mathrm{rank}(\mathcal{L}_Y) = 1.
\end{aligned}
$$

Let us compute the line $k = n$ (last line) of $G_{\alpha\beta}$ at a point $y \in Y \cap (U_\alpha \cap U_\beta)$,

$$\frac{\partial z_{\alpha,n}}{\partial z_{\beta,l}}(y) = \frac{\partial (h_{\alpha\beta} z_{beta,u})}{\partial z_{\beta,l}}(y) = \left(\frac{\partial h_{\alpha\beta}}{\partial z_{\beta,l}}\right) \underbrace{z_{\beta,n}(y)}_{=0} + h_{\alpha\beta}(y)\delta_{nl} = h_{\alpha\beta}(y)\delta_{nl},$$

so

$$G_{\alpha\beta} = \left(\begin{array}{c:c} g_{\alpha\beta} & * \\ \hdashline 0\dots 0 & h_{\alpha\beta} \end{array}\right)$$

with $g_{\alpha\beta}(x) \in GL_{n-1}(\mathbb{C}), h_{\alpha\beta}(x) \in \mathbb{C}^*$. This is of the form $G_{\alpha\beta} \leftrightarrow TX\big|_Y$. It follows that $\det(G_{\alpha\beta}\big|_Y) = \det(g_{\alpha\beta})\ h_{\alpha\beta}$ and hence $\det(TX\big|_Y) \cong \det(TY) \otimes \mathcal{L}_Y$. In addition, from adjunction one sees that $\det(TY) \otimes \mathcal{N}_{Y/X} \cong \det(TX\big|_Y)$ and thus it follows that $\mathcal{L}_Y \cong \mathcal{N}_{Y/X}$. □

Now, consider the normal/canonical bundle sequence; We want to study it for $Y^{(n-1)} \hookrightarrow \mathbb{P}^n$. In view of the result above one has:

$$0 \longrightarrow TY \longrightarrow TX\big|_Y \longrightarrow \mathcal{O}_X(D) \longrightarrow 0 \quad \longleftrightarrow \quad 0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow T^*X\big|_Y \longrightarrow T^*Y \longrightarrow 0$$

Projective Hypersurfaces: We know that $Y \overset{i}{\hookrightarrow} \mathbb{P}^n$ is given by the zero locus of a global section of a line bundle on $\mathbb{P}^n$. We can thus observe the following:

1. $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$ with $[\mathcal{O}_{\mathbb{P}^n}(k)] \mapsto k \in \mathbb{Z}$,
2. $H^0(\mathcal{O}_{\mathbb{P}^n}(k)) \cong \begin{cases} \mathbb{C}[x_0 \dots x_n]_{(k)} & \text{if } k \geq 0 \\ 0 & \text{if } k < 0, \end{cases}$ where global sections are contained in the first case.

This is enough to identify $\mathcal{L}_Y$ for $Y \overset{i}{\hookrightarrow} \mathbb{P}^n$ with $\mathcal{L}_Y \cong \mathcal{O}_{\mathbb{P}^n}(k)$ where $k > 0$. In other words, if $s \in H^0(\mathcal{O}(k))$ for $k > 0$, then we have a hypersurface $Y = \{s = 0\}$.

Example 16. Consider $Y := \{[X : Y : Z] \in \mathbb{P}^2 \mid X^2Y + Z^3 = 0\}$ and define $F := X^2Y + Z^3 \in H^0(\mathcal{O}_{\mathbb{P}^2}(3))$. Notice this defines a $g = 1$ curve in $\mathbb{P}^2$, actually an elliptic curve/complex torus. This follows from the genus-degree formula (see later) for $Y^d \subseteq \mathbb{P}^n \rightsquigarrow H^1(\mathcal{O}_Y) = \binom{d-1}{n}$.

1. global ÷ local: Let us dehomogenise the polynomial in $U_z = \{[X : Y : Z] \in \mathbb{P}^2 \mid Z \neq 0\} \cong \mathbb{C}^2$. Considering $f_z(u,v) := F(\frac{X}{Z}, \frac{Y}{Z}, 1)$ where $u := \frac{X}{Z}, v := \frac{Y}{Z}$ in $\mathbb{C}^2$. Then we have $f_z(u,v) = u^2v + 1 \subset \phi_z(U_z) \cong \mathbb{C}^2$. By the implicit function theorem $f_z^{-1}(0)$ is a complex manifold of dimension 1 and $(U_z, f_z(u,v) = u^2v + 1)$ is a local equation for Y.

2. local ÷ global: Changing coordinates via the trivialisations one has

$$\phi_{xz} = \phi_x \circ \phi_z^{-1}\left([X : Y : Z], \left(\frac{X}{Z}\right)^2 \frac{Y}{Z} + 1\right) = \phi_x\big(\underbrace{[X : Y : Z : X^2Y + Z^3]}_{\in \mathbb{P}^3 \setminus \{[0:0:0:1]\} \cong \mathbb{P}^2}\big)$$
$$= \left([X : Y : Z], \frac{Y}{X} + \left(\frac{Z}{X}\right)^3\right)$$

But then one sees that $\frac{Y}{X}+(\frac{Z}{X})^3=(\frac{Z}{X})^3[(\frac{X}{Z})^2\frac{Y}{Z}+1]$ and hence $f_x=(g_{xz})f_z,\ g_{xz}=(\frac{Z}{X})^3$ and $g_{xz}=(\frac{Z}{X})^3$ are the transition functions for $\mathcal{O}_{\mathbb{P}^2}(3)$ which identify F as a global section in $\mathcal{O}_{\mathbb{P}^2}(3)$.

$$\begin{cases} \text{patch local } (U_j, f_j) & \rightsquigarrow \text{global } F \in H^0\left(\mathcal{O}_{\mathbb{P}^n}(k)\right) \\ \text{restrict global } F \in H^0\left(\mathcal{O}_{\mathbb{P}^n}(k)\right) & \rightsquigarrow \text{local } (U_j, f_j) \end{cases}$$

Given the discussion above we have immediately the following:

Corollary. If $Y \overset{i}{\hookrightarrow} \mathbb{P}^n$ is a hypersurface of codimension 1, then

$$\mathcal{N}_{Y/\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(d)\big|_Y \tag{63}$$

where d is the degree of the hypersurface $Y \hookrightarrow Y$.

Proof. Simply $\mathcal{L}_Y \cong \mathcal{O}_{\mathbb{P}^n}(d)$ and $\mathcal{N}_{Y/X} \cong \mathcal{L}_Y\big|_Y \cong i * \mathcal{O}_{\mathbb{P}^n}(d) \equiv \mathcal{O}_{\mathbb{P}^n}(d)\big|_Y$. □

Corollary (Adjunction)**.** The following holds true:

1. $K_{\mathbb{P}^n}(=\bigwedge^n T^*_{\mathbb{P}^n}) \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$.

2. For $Y \hookrightarrow \mathbb{P}^n$ a hypersurface of codimension 1 and degree d, one has the *adjunction formula*:

$$K_Y \cong \mathcal{O}_{\mathbb{P}^n}(d-n-1)\big|_Y. \tag{64}$$

Proof. Starting with the first statement, just take *det* from $0 \to T^*_{\mathbb{P}^n} \to \mathcal{O}^{\oplus n+1}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to 0$: $\det(\mathcal{O}^{\oplus n+1}_{\mathbb{P}^n}(-1)) \cong \det(T^*_{\mathbb{P}^n}) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \det(\mathcal{O}_{\mathbb{P}^n}) \cong \det(T^*_{\mathbb{P}^n})$. Since $\det(\mathcal{O}_{\mathbb{P}^n}) \cong \mathcal{O}_{\mathbb{P}^n}$ and $\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n} \cong \mathcal{F}$ for all sheaves $\mathcal{F}$ of $\mathcal{O}_{\mathbb{P}^n}$-modules. Also $\det(\mathcal{O}^{\oplus n+1}_{\mathbb{P}^n}(-1)) \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$. It follows that $K_{\mathbb{P}^n} \cong \det(T^*_{\mathbb{P}^n}) \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$.

For the second statement, from the general adjunction formula one has $K_Y \cong K_X\big|_Y \otimes \mathcal{N}_{Y/X}$. For $X=\mathbb{P}^n$ and $Y \overset{i}{\hookrightarrow} \mathbb{P}^n$ of dimension $n-1$, one has $K_X \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$ and $\mathcal{L}_Y \cong \mathcal{O}_{\mathbb{P}^n}(d)\big|_Y$ for d the degree of Y. Hence $K_Y \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)\big| \otimes \mathcal{O}_{\mathbb{P}^n}(d)\big|_Y \cong \mathcal{O}_{\mathbb{P}^n}(d-n-1)\big|_Y$. □

Example 17 (Quintic in $\mathbb{P}^4$)**.** Consider

$$Y_3 := \left\{ [X] \in \mathbb{P}^4 \;\middle|\; \sum_{i=1}^4 X_i^5 + c\prod_{i=0}^4 X_i = 0,\ c \in \mathbb{C} \right\}. \tag{65}$$

Then $K_{Y_3} \cong \mathcal{O}_{\mathbb{P}^4}(5-4-1)\big|_{Y_3} = \mathcal{O}_{\mathbb{P}^4}(0)\big|_{Y_3} \equiv \mathcal{O}_{Y_3}$. This means Y_3 is a *Calabi-Yau 3-fold*! Superstrings in $D=10$ compactify on Y_3: $\mathbb{R}^{10} \cong \mathbb{R}^4 \times Y_3$ where there is the effective theory and $\mathcal{N}=1$ SUSY on $\mathbb{R}^4$ and Y_3 is compact.

6.4 Ideal Sheaf Sequence and Degree-Genus-Formula

To any complex submanifold $Y \overset{i}{\hookrightarrow} X$ is attached a short exact sequence:

$$0 \longrightarrow j_Y \longrightarrow \mathcal{O}_X \xrightarrow{i^*} i^*\mathcal{O}_Y \longrightarrow 0$$

This is the *ideal sheaf sequence.* Note that this is a sequence of sheaves on X. Indeed, $U \supseteq X \longmapsto i_*\mathcal{O}_Y(U) = \mathcal{O}_Y(i^{-1}(U))$. Also, notice the following:

1. $X \supseteq U \longmapsto j_Y(U) := \{f : U \to \mathbb{C} \mid f \text{ holomorphic}, \ f(U \cap Y) = 0\}$, so f is in $\mathcal{O}_X(U)$ and vanishing along $Y \hookrightarrow X$. This is a sheaf of ideals inside $\mathcal{O}_X$.

2. $i_*\mathcal{O}_Y := \mathcal{O}_X / j_X$, alternatively $j_X := \ker(i^* : \mathcal{O}_X \to i_*\mathcal{O}_Y)$.

Codimension 1 hypersurface in $\mathbb{P}^n$: In this case one has $Y \hookrightarrow \mathbb{P}^n$:

$$0 \longrightarrow j_Y \xrightarrow{\cdot F} \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

where $\cdot F$ is the multiplication by the defining equation of $Y = \{F = 0\}$ (F is a homogeneous polynomial). **Important:** In this case one has $j_Y \cong \mathcal{O}_{\mathbb{P}^n}(-d)$ where d is the degree of F.

Degree-Genus-Formula: One can find a relation between the degree of F and the genus g of the associated plane curve $\mathcal{C} \hookrightarrow \mathrm{Pr}^2$.

Definition 34 (Genus of X)**.** We define the (arithmetic) *genus* of a complex projective manifold of dimension n as

$$g := (-1)^n \big(\chi(\mathcal{O}_X) - 1\big) = (-1)^n \left(\sum_{l=0}^{n} (-1)^l \dim H^l(X, \mathcal{O}_X) - 1 \right). \tag{66}$$

Remark. If $\mathcal{C}$ is of dimension 1 and projective:

$$g = -\big(\dim H^0(\mathcal{C}, \mathcal{O}_\mathcal{C}) - \dim H^1(\mathcal{C}, \mathcal{O}_X) - 1\big) = \dim H^1(\mathcal{O}_X) \tag{67}$$

Setting: $\mathcal{C} \overset{i}{\hookrightarrow} \mathbb{P}^2$ defined by $F = 0$ with $F \in H^0(\mathbb{P}^2, \mathcal{O}(d))$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow i_*\mathcal{O}_\mathcal{C} \longrightarrow 0$$

Then this induces a long exact sequence in cohomology:

1. $0 \longrightarrow \underbrace{H^0(\mathcal{O}(-d))}_{\cong 0} \longrightarrow \underbrace{H^0(\mathcal{O}_{\mathbb{P}^2})}_{\cong \mathbb{C}} \xrightarrow{i} H^0(i_*\mathcal{O}_\mathcal{C}) \cong \underbrace{H^0(\mathcal{O}_\mathcal{C})}_{\cong \mathbb{C}} \longrightarrow \underbrace{H^1(\mathcal{O}(-d))}_{\cong 0} \longrightarrow \dots$
   This says that we have $0 \longrightarrow \mathbb{C} \xrightarrow{\cong} \mathbb{C} \longrightarrow 0$.

2. $0 \longrightarrow \underbrace{H^1(\mathcal{O}_{\mathbb{P}^2})}_{\cong 0} \longrightarrow H^1(\mathcal{O}_\mathcal{C}) \longrightarrow H^2(\mathcal{O}_{\mathbb{P}^2}(-d)) \longrightarrow \underbrace{H^2(\mathcal{O}_{\mathbb{P}^2})}_{\cong 0} \longrightarrow \dots$
   This says that we have $H^1(\mathcal{O}_\mathcal{C} \cong H^2(\mathcal{O}_{\mathbb{P}^2}(-d))$. We conclude that $h^1(\mathcal{O}_\mathcal{C}) = h^2(\mathcal{O}_{\mathbb{P}^2}(-d)) = \binom{d-1}{d-2-1} = \binom{d-1}{(d-1)-2} = \binom{d-1}{2}$ and hence $g\big(\mathcal{C} \hookrightarrow \mathbb{P}^2\big) = \binom{d-1}{2} = \frac{1}{2}(d-1)(d-2)$.

Example 18. Consider the following three examples:

1. $\mathcal{C}_1 := \{F = X_0^2 + X_1X_2 = 0\} \subseteq \mathbb{P}^2$ has $g(\mathcal{C}_1) = 0$ which implies $\mathcal{C}_1 \cong \mathbb{P}^1$.

2. $\mathcal{C}_2 := \{F = X_0^3 + X_1^3 + X_2^3 = 0\} \subseteq \mathbb{P}^2$ has $g(\mathcal{C}_2) = 1$ and hence $\mathcal{C}_2 \cong E$, a torus!

3. $\mathcal{C}_3 := \{F = X_0^4 + X_1^4 = 0\} \subseteq \mathbb{P}^2$ has $g(\mathcal{C}_3) = 3$.

Question: Where are genus 2 curves?

Hyperelliptic Curves: Consider $y^2 = p(x)$ where $P(x) \in \mathbb{C}[x]$, $\deg(P) = 2g + 1 + \varepsilon$ with $\varepsilon \in \{0, 1\}$ and distinct roots. This means $y^2 = \prod_i^{2g+1+\varepsilon}(x - r_i)$ where the r_i are the roots, i.e. $P(r_i) = 0$.

1. Note that $y^2 = P(x)$ is an affine plane curve in $\mathbb{C}^2$, $(x, y) \in \mathbb{C}^2$, we call it X.

2. $U = \{(x, y) \in X \text{ with } x \neq 0\}$ is an open set for $X \subseteq \mathbb{C}^2$.

3. Let $Q(z) = z^{2g+2}P(\frac{1}{z})$: This is a polynomial in z with distinct roots (since P has distinct roots).

4. $w^2 = Q(z)$ is an affine plane curve in $\mathbb{C}^2$, we call it Y.

5. $V = \{(z, w) \in Y \text{ with } z \neq 0\}$ is an open set for Y in $\mathbb{C}^2$.

$$\mathbb{C}^2 \supseteq X \equiv \{y^2 - P(x) = 0\} \qquad \{w^2 - Q(z) = 0\} \equiv Y \subseteq \mathbb{C}^2$$
$$U \supseteq X \overset{\text{glue}}{\leftrightsquigarrow} V \subseteq Y$$

with gluing via

$$\begin{aligned} \varphi : U &\longrightarrow V \\ (x, y) &\longmapsto (z, w) = \left(\frac{1}{x}, \frac{y}{x^{g+1}}\right). \end{aligned}$$

6. The surface $X \coprod Y/\tilde{\varphi}$ obtained via this gluing is a compact Riemann surface of genus g and is called *hyperelliptic.*

Genus 2: It turns out that all $g = 2$ compact Riemann surfaces are hyperelliptic, e.g. $y^2 = x(x - 1)(x - 2)(x - 3) \subseteq \mathbb{C}^2$ and gluing. These particular curves exist at every genus g and they can be seen geometrically as given by a ramified double covering $\pi : \mathcal{C} \xrightarrow{2:1} \mathbb{P}^1$. The ramification points occur at the roots of $P(x)$. If $P(x)$ is of odd degree, it is also ramified at $p = \{\infty\}$.

6.5 Relations in Cohomology: Serre Duality

Serre duality is one of the most fundamental relations between cohomology groups of a certain sheaf and its dual: this relation is mediated by the canonical sheaf $\Omega_X^n \equiv K_X$. This is one of the many reasons why the canonical sheaf is so important!

The duality states

$$H^i(X, \mathcal{F}) \cong H^{n-1}(X, \mathcal{F}^* \otimes K_X)^*. \tag{68}$$

Here $n = \dim_{\mathbb{C}} X$ and $\mathcal{F}^* = \hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ is the dual of $\mathcal{F}$. Note that $\mathcal{F} = \mathcal{O}_X$ on a curve: $H^1(\mathcal{O}_X) \cong H^0(X, \Omega^1_X)^*$ which gives the genus.

Remark. One has to look at this as a "perfect paring":

$$H^i(X, \mathcal{F}) \times H^{n-i}(X, \mathcal{F}^* \otimes K_X) \overset{\text{non-deg}}{\longrightarrow} \mathbb{C}$$

Let us compare this to *Poincaré duality*: For a compact smooth manifold M it states

$$\begin{aligned} H^i_{\text{dR}}(M) \times H^{n-i}_{\text{dR}}(M) &\overset{\text{n.d.}}{\longrightarrow} \mathbb{R} \\ (\omega, \eta) &\longmapsto \int_M \omega \wedge \eta, \end{aligned} \tag{69}$$

so $H^i_{\text{dR}}(M) \cong H^{n-i}_{\text{dR}}(M)^*$. Note that $\dim V = \dim V^*$ for any vector space.

7 Compact Riemann Surfaces

Compact *Riemann Surfaces* are compact complex manifolds of dimension 1, hence locally they are described by a single coordinate function $z : U \to \mathbb{C}$ for $U \subseteq \mathcal{C}$. Their geometry is very special (and beautiful).

Remark. Obviously, closed strings are modelled by compact Riemann surfaces.

7.1 Setting the Stage

Topology: The topology of compact Riemann surfaces is very easy and fully characterised by a single invariant, the genus g.

Note that $H^i(\mathcal{C}, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0, 2 \\ \mathbb{Z}^{2g} & i = 1 \end{cases}$.

Figure 2: Examples for compact Riemann surfaces for $g = 0$ (left) and $g = 1$ (right)

This also gives a very important information:

$$\begin{aligned} \mathcal{C}_1 : H^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}}) &\overset{\delta}{\longrightarrow} H^2(\mathcal{C}, \mathbb{Z}) \cong \mathbb{Z} \\ [\mathcal{L}] &\longmapsto \mathcal{C}_1\big([\mathcal{L}]\big) = n \end{aligned}$$

Line bundles are classified by $\mathcal{C}_1$ which is a discrete number. In this context, this map is called the *degree* of the line bundle:

$$\deg(\mathcal{L}) := \mathcal{C}_1(\mathcal{L}) \in \mathbb{Z} \tag{70}$$

Example 19. $\deg(\mathcal{O}_{\mathbb{P}^1}(k)) = k \in \mathbb{Z}$.

Remark (Numerical Criterion)**.** One can establish some results regarding the relation between the cohomology and the degree of a line bundle:

$$\deg(\mathcal{L}) < 0 \implies H^0(\mathcal{C}, \mathcal{L})$$

Intuitively, $\deg(\mathcal{L}) = (\#\text{zeros}) - (\#\text{poles})$ of a section!

Theorem 24 (Riemann-Roch)**.** *Let $\mathcal{C}$ be a compact Riemann surface and let $\mathcal{L}$ be a line bundle on it. Then*

$$h^0\big(\mathcal{C}, \mathcal{L}\big) - h^1\big(\mathcal{C}, \mathcal{L}\big) = 1 - g + \deg\big(\mathcal{L}\big) \tag{71}$$

where $h^i = \dim H^i$.

Note. The theorem establishes a relation between the cohomology groups of a line bundle on a compact Riemann surface. It is one of the most useful results in complex algebraic geometry!

Corollary. Let $K_{\mathcal{C}} = T^*_{\mathcal{C}}$, the canonical bundle on $\mathcal{C}$ (i.e. the bundle of holomorphic 1-forms). Then one has that

$$\deg\big(\mathcal{K}_{\mathcal{C}}\big) = 2g - 2. \tag{72}$$

Proof. Recall that $g = h^1(\mathcal{O}_{\mathcal{C}}) = h^0(K_{\mathcal{C}})$ by Serre duality. Then $\underbrace{h^0(K_{\mathcal{C}})}_{g} - h^1(K_{\mathcal{C}}) = 1 - g + \deg(K_{\mathcal{C}})$. Using Serre duality, $h^1(K_{\mathcal{C}}) = h^0(T_{\mathcal{C}} \otimes K_{\mathcal{C}})$, but $T_{\mathcal{C}} \otimes K_{\mathcal{C}} = T_{\mathcal{C}} \otimes T_{\mathcal{C}}* \cong \mathcal{O}_{\mathcal{C}}$. It follows that $h^0(K_{\mathcal{C}}) - h^0(\mathcal{O}_{\mathcal{C}}) = g - 1$ and hence $\deg(K_{\mathcal{C}}) = 2g - 2$. □

Remark (Dimension of the Moduli Space $\mathcal{M}_g$)**.** For a complex manifold X we have seen that $H^0(T_X)$ is related to the automorphisms, that is those maps that preserve a certain (complex) structure. $H^1(T_X)$ can be interpreted as a sort of "defect": It tells how much a certain complex structure can change (without changing the topology!). More precisely, $H^1(T_X)$ gives a very rough representation of the *moduli space of complex structures* on X, namely $H^1(T_X) \cong T_{[X]}\mathcal{M}$. Nonetheless this is enough to compute the dimension!

Dimension of $\mathcal{M}_{\geq 2}$ We use Riemann-Roch to compute $h^1(T_{\mathcal{C}})$:

1. Serre duality: $h^1(T_{\mathcal{C}}) = h^0(K_{\mathcal{C}}^{\otimes}) \rightsquigarrow$ holomorphic quadratic differentials.
2. Riemann-Roch: $h^0(K_{\mathcal{C}}^{\otimes 2}) - h^1(K_{\mathcal{C}}^{\otimes 2}) = 2(2g-2) - g + 1 = 3g - 3$.

3. $h^1(K_{\mathcal{C}}^{\otimes}) = h^0(T_{\mathcal{C}}^{\otimes} \otimes K_{\mathcal{C}}) = h^0(T_{\mathcal{C}})$ but $\deg(T_{\mathcal{C}}) = -\deg(K_{\mathcal{C}}) = 2-2g$ and if $g \geq 2$, then $2-2g < 0$ which implies $h^0(T_{\mathcal{C}}) = h^1(K_{\mathcal{C}}^{\otimes 2}) = 0$

It follows that $h^1(T_{\mathcal{C}}) = 3g-3$ if $g \geq 2$. Some examples:

Consider the Riemann sphere ($g = 0$). Here $h^1(T_{\mathbb{P}^1}) = h^1(\mathcal{O}_{\mathbb{P}^1}(+2)) = 0$, no moduli! (The moduli space is a "point" plus isomorphisms.)

Next, consider tori/elliptic curves ($g = 1$). Here $h^1(T_{\mathbb{E}}) = h^1(\mathcal{O}_{\mathbb{E}}) \overset{\text{S.D.}}{=} h^0(\mathcal{O}_{\mathbb{E}}) = 1$.

To conclude:

$$\dim_{\mathbb{C}} \mathcal{M}_g = \begin{cases} 0, & g = 0, \\ 1, & g = 1, \\ 3g-3, & g \geq 2 \end{cases}$$

Definition 35 (Hodge Numbers)**.** Let X be a compact complex manifold. Then we call $h^{p,q}(X) := \dim H^q(X, \Omega_X^p)$ the *Hodge numbers* of X.

Remark. The numbers $h^{p,q}(X)$ can be arranged into a "diamond"-shaped figure, the *Hodge diamond*. For example, consider $\dim_{\mathbb{C}} X = 2$:

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{3,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

Figure 3: Hodge diamond for $\dim_{\mathbb{C}} X = 2$.

Note. Not all the $h^{p,q}$ are independent! They are related by symmetries:

1. Hodge symmetry: $h^{p,q}(X) = h^{q \cdot p}(X)$.

2. Serre duality: $h^{p,q}(X) = h^{n-p,n-q}(X)$. Indeed, $H^q(\Omega_X^p) \cong H^{n-q}(\Omega_X^n \otimes \bigwedge^p T_X) \cong H^{n-q}(\Omega_X^{n-p})$ by using the pairing.

Hodge diamond and topology: Let us now look at the Hodge diamond of a compact Riemann surface of genus g:

$$h^{1,1} = h^1(\Omega^1_{\mathcal{C}}) = 1$$

$$h^{1,0} = h^0(\Omega^1_{\mathcal{C}}) = g \qquad\qquad h^{0,1} = h^1(\mathcal{O}_{\mathcal{C}}) = g$$

$$h^{0,0} = h^0(\mathcal{O}_{\mathcal{C}}) = 1$$

One can observe that the sum of the Hodge numbers on the rows give the *Betti numbers* $b^(\mathcal{C})$ of $\mathcal{C}$, indeed:

$$b^i(\mathcal{C}) = \dim\left(H^i_{\mathrm{dR}}(\mathcal{C}) \otimes \mathbb{C}\right) = \dim\left(H^i(\mathcal{C}, \mathbb{Z}) \otimes \mathbb{C}\right) = \begin{cases} 1, & i = 0, 2 \\ 2g, & i = 1 \end{cases} = \sum_{p+q=i} h^{p,q}(\mathcal{C})$$

In fact, this is a very general and important result:

Theorem 25 (Hodge Theorem)**.** *Let X be a compact connected (Kähler) manifold. Then*

$$H^i(X, \mathbb{C}) \cong \bigoplus_{p+q=i} H^q\left(X, \Omega^p_X\right) \tag{73}$$

where $H^i(X, \mathbb{C}) = H^i_{dR}(X) \otimes \mathbb{C}$ is the de Rham-cohomology valued in $\mathbb{C}$.

7.2 Moduli Space of Genus 1 Compact Riemann Surfaces

We say that (w_1, w_2) such that $\Lambda = \mathrm{span}_{\mathbb{Z}}(w_1, w_2)$ for linearly independent $w_1, w_2 \in \mathbb{C}$ over $\mathbb{R}$ determines the complex structure of $\mathbb{E} = \mathbb{C}/\Lambda$. Recall that $\Lambda \equiv \Lambda(w_1, w_2) := \{nw_1 + mw_2 \mid n, m \in \mathbb{Z}\}$.

Question: When do pairs (w_1, w_2) and $(\tilde{w}_1, \tilde{w}_2)$ determine the same complex structure?

Remark. Without loss of generality we can assume $\mathrm{Im}(\frac{w_2}{w_1}) > 0$ and $\mathrm{Im}(\frac{\tilde{w}_2}{\tilde{w}_1}) > 0$.

Lemma 6. *We have the following equivalence:*

$$\Lambda(w_1, w_2) = \Lambda(\tilde{w}_1, \tilde{w}_2) \Longleftrightarrow \exists A \in PSL(2, \mathbb{Z}) := SL(2, \mathbb{Z})/\{\pm 1\} : \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} = A \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Proof. Let us prove the two implications separately:

" $\Longleftarrow$ ": Suppose $\underline{\tilde{w}} = A\underline{w}$ for $A \in PSL(2, \mathbb{Z})$. Then $\underline{\tilde{w}} \in \Lambda(\underline{w})$ and hence it follows that $\Lambda(\underline{\tilde{w}}) \subseteq \Lambda(\underline{w})$. Conversely, suppose $\underline{w} = A^{-1}\underline{\tilde{w}}$, hence $\Lambda(\underline{w}) \subseteq \Lambda(\underline{\tilde{w}})$. It follows that $\Lambda(\underline{\tilde{w}}) = \Lambda(\underline{w})$.

" $\Longrightarrow$ ": Let $\Lambda(\underline{w}) = \Lambda(\underline{\tilde{w}})$. This means that $\underline{\tilde{w}} \in \Lambda(\underline{w})$ and $\underline{w} \in \Lambda(\underline{\tilde{w}})$. Therefore $\underline{\tilde{w}} = A\underline{w}$ and $\underline{w} = \tilde{A}\underline{\tilde{w}}$. Hence on has

$$\underline{w} = \tilde{A}\underline{\tilde{w}} = \tilde{A}A\underline{w} \implies \tilde{A}A = \mathbb{1}_2 \implies \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, from $\det(\tilde{A}A) = \det(\tilde{A})\det(A) = 1$ one has $(\tilde{a}\tilde{d} - \tilde{c}\tilde{b})(ad - bc) = 1$. Since $a, b, c, d \in \mathbb{Z}$, this is only possible if $ad - bc = \pm 1$. Now consider the following: $\tilde{w}_2 = cw_1 + dw_2$, $\tilde{w}_1 = aw_1 + bw_2$. Then, defining $\tau := \frac{w_2}{w_1} = x + iy$ with $x, y \in \mathbb{R}$, we calculate:

$$\begin{aligned}\frac{\tilde{w}_2}{\tilde{w}_1} &= \frac{cw_1 + dw_2}{aw_1 + bw_2} = \frac{d\frac{w_2}{w_1} + c}{b\frac{w_2}{w_1} + a} = \frac{d\tau + c}{b\tau + a} = \frac{(d\tau + c)(b\bar{\tau} + a)}{|b\tau + a|^2} \\ &= \frac{1}{|b\tau + a|^2}\Big[\big(d(x+iy) + c\big)\big(b(x - iy) + a\big)\Big] \\ &= \frac{1}{|b\tau + a|^2}\big(bdx^2 + daxcbx + ca + bdy^2 + i(ad - bc)y\big)\end{aligned}$$

Now using that $0 < \mathrm{Im}(\tau) = y$, we have

$$0 < \mathrm{Im}\left(\frac{\tilde{w}_2}{\tilde{w}_1}\right) = \frac{ad - bc}{|b\tau + a|^2}\,\mathrm{Im}\Big(\underbrace{\frac{w_2}{w_1}}_{>0}\Big)$$

which implies $ad - bc = 1$ and hence $A \in SL(2, \mathbb{Z})$. Finally, notice that A and $-A$ maps to the same lattice $\tilde{\Lambda}$ and thus one has to identify them. This leads to $PSL(2, \mathbb{Z})$.

□

Theorem 26. $\mathbb{E} = \mathbb{C}/\Lambda(\underline{w})$ *has the same complex structure as* $\tilde{\mathbb{E}}$ *if and only if there exists* $A \in PSL(2, \mathbb{Z})$ *and* $\lambda \in \mathbb{C}^\times$ *such that* $\underline{\tilde{w}} = \lambda A \underline{w}$.

Proof. Again, we prove the implications separately:

"$\Longrightarrow$": Assume $\mathbb{E} \cong \tilde{\mathbb{E}}$. This means that there exists a biholomorphic map $\mathbb{C}/\Lambda(w) \xrightarrow{\varphi} \mathbb{C}/\Lambda(\tilde{w})$ that can be lifted to the universal covering of $\mathbb{E}$ and $\tilde{\mathbb{E}}$. Namely:

$$\begin{array}{ccc}\mathbb{C} & \xrightarrow{\quad h \quad} & \mathbb{C} \\ \big\downarrow \pi & & \big\downarrow \hat{\pi} \\ \mathbb{C}/\Lambda(w) & \xrightarrow{\quad \varphi \quad} & \mathbb{C}/\Lambda(\tilde{w})\end{array}$$

One can choose $0 \in \mathbb{C}$ and define $h : \mathbb{C} \to \mathbb{C}$ such that $\hat{\pi} \circ h(0) = \varphi \circ \pi(0)$. Now, this holds true locally around the origin and it gives a biholomorphic map $U_{p=0} \xrightarrow{h_0} \tilde{U}_{p=0}$ for neighbourhoods $U, \tilde{U} \subseteq \mathbb{C}$ of the origin. By analytic continuation $h_0 \rightsquigarrow h : \mathbb{C} \to \mathbb{C}$ biholomorphic such that $\pi \circ h = \varphi \circ \pi$ everywhere in $\mathbb{C}$.

On the other hand $h \in \mathrm{Aut}(\mathbb{C})$ are well-known: They are of the form $h : \mathbb{C} \to \mathbb{C},\ z \mapsto az + b$ with $a \in \mathbb{C}^\times,\ b \in \mathbb{C}$. This means that

$$w_1 \xmapsto{h} az + b \xmapsto{\hat{\pi}} aw_1 + b + \Lambda_2 \quad \text{and} \quad w_2 \xmapsto{\pi} w_1 + \Lambda_1 = \Lambda_1 \xmapsto{\varphi} b + \Lambda_2$$

and since one has $\hat{\pi}h = \varphi\pi$ it follows that $aw_1 \in \Lambda_2$.

Remark: Note that in general $\varphi(z+\Lambda_1) = h(z) + \Lambda_2$ from the theory of universal coverings. But then, since $dz = \lambda_i \in H^0(\mathbb{C}/\Lambda_i, \Omega^1_{\mathbb{C}/\Lambda_i})$, one has that $\varphi^*\lambda_2 = a\lambda_1$ by changing basis. Hence $\varphi^*\lambda_2 = adz$. On the other hand $\varphi^*\lambda_2 0dh = \partial_z h dz$ if h is biholomorphic. Then one gets a differential equation $\partial_z h = a$, so $\int_0^z \partial_w h dw = \int_0^z a dw$ and we obtain $h(z) = az + b$ for $b \in \mathbb{C}$ such that $h(0) = b$. Note that $h(0) = b \in \mathbb{C}$ is just an overall translation. One might require $h(0) = 0$, i.e. 0 is mapped to 0.

Clearly, the same is true for going from $\underline{\tilde{w}}$ to $\underline{w}$ via h: one finds that $\tilde{a}\underline{\tilde{w}} \in \Lambda_1$. It follows that if $\mathbb{E} \cong \tilde{\mathbb{E}}$, then $\underline{\tilde{w}} = aA\underline{w}$ with $a \in \mathbb{C}^\times$ and $A \in PSL(2,\mathbb{Z})$.

"$\Longleftarrow$": We already showed that $\underline{w}$ and $A\underline{w}$ define the same lattice up to translations. This is just a change of basis in the lattice. To account for the translation we consider $h(z) = z + b$. Similarly also $\underline{w}$ and $a\underline{w}$ define the same lattice. this amounts to consider $h(z) = az + b$.

□

Remark (Long Story Short)**.** $A \in PSL(2,\mathbb{Z})$ is a change of basis of the lattice: As it is natural it does not change the complex structure. On the other hand one can directly observe that one has the isomorphisms

$$\mathbb{E}_1 \ni [z] = z + \left(n \cdot 1 + m\frac{w_2}{w_1}\right) \xleftrightarrow[1:1]{\varphi} w_1 z + \left(nw_1 + mw_2\right) = \varphi\big([z]\big).$$

Hence it is enough to consider lattices generated by the following pair: $\Lambda = \mathrm{span}_{\mathbb{Z}}(1,\tau)$ where $\tau = \frac{w_2}{w_1}$ and $\mathrm{Im}(\tau) > 0$. Important: This allows to restrict to consider the *Poincaré Half-Plane*:

$$\mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\} \tag{74}$$

where τ is the *modulus.*

Idea of Moduli Space: Take a suitable quotient of $\mathbb{H}$ so that each complex structure induced by a lattice is contained only once:

$$\mathcal{M}_{g=1} \cong \mathbb{H}/PSL(2,\mathbb{Z})$$

with $PSL(2,\mathbb{Z})$ the modular group. First of all, notice that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w_2 \\ w_1 \end{pmatrix} = \begin{pmatrix} aw_2 + bw_1 \\ cw_2 + dw_1 \end{pmatrix}$, then

$$\tau = \frac{w_2}{w_1} \longmapsto \frac{aw_2 + bw_1}{cw_2 + dw_1} = \frac{1\tau + b}{c\tau + d}$$

which is a *fractional linear transformation* since $ad - bc = 1$.

Definition 36 (Fundamental Domain)**.** It is $z_1 \sim z_2$ in $\mathbb{H}$ if there exists $g \in PSL(2,\mathbb{Z})$ such that $z_2 = gz_1$ (i.e. z_2 is in the orbit). A *fundamental domain* for $PSL(2,\mathbb{Z})$ is an open set $\mathcal{D} \subseteq \mathbb{H}$ which does not contain any points of distinct equivalent points and such that $\overline{\mathcal{D}}$ (point set closure) contains at least one point from each equivalence class.

It follows from this that the orbit of $\mathcal{D}$ covers $\mathbb{H}$.

Remark. Finding $\mathcal{M}_{g=1}$ is "the same" as finding $\mathcal{D}$ for $PSL(2,\mathbb{Z})$ in $\mathbb{H}$.

Lemma 7. *Let $z \in \mathbb{H}$ be arbitrary but fixed. Then, there is only a finite number of $(c,d) \in \mathbb{Z}^2$ such that $|cz+d| \leq 1$.*

Proof. Let (c,d) be such that $|cz+d| \leq 1$. Then, posing $z = x+iy$ we have $|cz+d|^2 = \underbrace{(cd+d)^2}_{\geq 0} + c^2y^2$ and thus $c^2y^2 \leq (cx+d)^2 + c^2y^2 \leq 1$. Since $z \in \mathbb{H}$ with $y > 0$ it follows that $|c| \leq \frac{1}{y}$. Now, since $c \in \mathbb{Z}$ there is only a finite number of points with this property. Then, let $\hat{c}$ be one of such values, i.e. $|\hat{c}| \leq \frac{1}{y}$. Regarding d, it is easy to see that $(\hat{c}x+d)^2 + \hat{c}^2y^2 \leq 1$ is only satisfied for a finite number of values of $d \in \mathbb{Z}$. □

Lemma 8. *Let $z \in \mathbb{H}$ be arbitrary but fixed and let $PSL(2,\mathbb{Z})$ act on z. Then there exists only a finite number of points $g \cdot z \in \mathbb{H}$ such that for any $g \in PSL(2,\mathbb{Z})$ we have $\operatorname{Im}(g \cdot z) > \operatorname{Im}(z)$.*

Proof. For any $g \in PSL(2,\mathbb{Z})$ and $z \in \mathbb{H}$ one has

$$g \cdot z = \frac{az+b}{cz+d} = \frac{az+b}{cz+d}\frac{c\bar{z}+d}{c\bar{z}+d} = \operatorname{Re}(g\cdot z) + i\frac{ad-bc}{|cz+d|^2}\operatorname{Im}(z)$$

and since $ad - bc = 1$ it follows that $\operatorname{Im}(g \cdot z) = \frac{\operatorname{Im}(z)}{|cz+d|^2}$. Finally, lemma 7 tells that there is only a finite number of pairs (c,d) such that $|cz+d| \leq 1$. □

Remark. The previous lemma 8 suggests that among the elements of an equivalence class $g \cdot z$ one can choose an element of *maximal height*, i.e. a representative such that $|cz+d| \geq 1$ for all $(c,d) \in \mathbb{Z}^2$:

$$[g\cdot z] \sim \hat{z} \equiv \hat{g}\cdot z \quad \text{for some } \hat{z} \in PSL(2,\mathbb{Z}): \ |c\hat{z}+d| \geq 1 \ \forall (c,d) \in \mathbb{Z}^2 \tag{75}$$

Remark. Also notice $g : z \longmapsto g \cdot z = z+1$ is a legit modular transformation in $PSL(2,\mathbb{Z})$ (just choose $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$). Then, every element in $\mathbb{H}$ will be mapped in the strip given by $-\frac{1}{2} \leq \operatorname{Re}(z) \leq \frac{1}{2}$:

$$[g\cdot z] \sim |z| \leq \frac{1}{2} \text{ for } g : z \longmapsto z+n \text{ with } n \in \mathbb{Z} \tag{76}$$

(with $g \cdot g(z) = z+2$, $g^{-1}(z) = z-1$)

Theorem 27. *The fundamental domain for the group $PSL(2,\mathbb{Z})$ is the set*

$$\mathcal{D} = \left\{ z \in \mathbb{H} \ \middle| \ |\operatorname{Re}(z)| < \frac{1}{2}, \ |z| > 1 \right\}. \tag{77}$$

In particular, there is a set theoretic isomorphism $\mathcal{M}_{g=1} \cong \mathcal{D}$.

Proof. First, we show that $\mathcal{D} \cong \{z \in \mathbb{H} \mid |\mathrm{Re}(z)| < \frac{1}{2},\ |cz+d| > 1\ \forall (c,d) \in \mathbb{Z}^2\}$. We call this set $\mathcal{D}_1$. Clearly $\mathcal{D}_1 \subseteq \mathcal{D}$ since if $z \in \mathcal{D}_1$, then for $c = 1$, $d = 0$ one has $|z| > 1$ and hence $z \in \mathcal{D}$.

Viceversa, suppose $z \in \mathcal{D}$. Then if $z = x + iy$ we have $|cz+d|^2 = (cx+d)^2 + c^2y^2 = c^2(\underbrace{x^2+y^2}_{>0}) + 2cdx + d^2$. Since $x = \mathrm{Re}(z) > -\frac{1}{2}$ we conclude $|cz+d|^2 > c^2 2cdx + d^2 > c^2 - cd + d^2 > 1$ if $(c,d) \neq (0,0)$. It follows that if $z \in \mathcal{D}$ then $z \in \mathcal{D}_1$, so that $\mathcal{D} = \mathcal{D}_1$.

Then, by the previous remark one has that $\overline{\mathcal{D}}$ contains at least one point from each equivalence class under $PSL(2,\mathbb{Z})$. In particular, the only pairs of points which are equivalent under $PSL(2,\mathbb{Z})$ are the points on the boundary ∂D of D which are mapped into another by a reflection about $x = 0$. Indeed, say $z \sim z'$ and $z' = g \cdot z$, then $\mathrm{Im}(z) = \mathrm{Im}(g \cdot z) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$ which implies $|cz+d|^2 \stackrel{!}{=} 1$. This is possible for the following choices:

$$c = \pm 1,\ d = 0 \rightsquigarrow z \longmapsto -\frac{1}{z},$$
$$c = 0,\ d = \pm 1 \rightsquigarrow z \longmapsto z + 1$$

Clearly the transformation $z \longmapsto z + 1$ maps the points with $\mathrm{Re}(z) = -\frac{1}{2}$ to $\mathrm{Re}(z') = \frac{1}{2}$. Further, if $|z| = 1$, then $z = e^{i\theta} \longmapsto -e^{-i\theta}$. This proves that the only points which are identified in $\overline{\mathcal{D}}$ are points in ∂D which coincides upon reflection about $x = 0$. $\square$

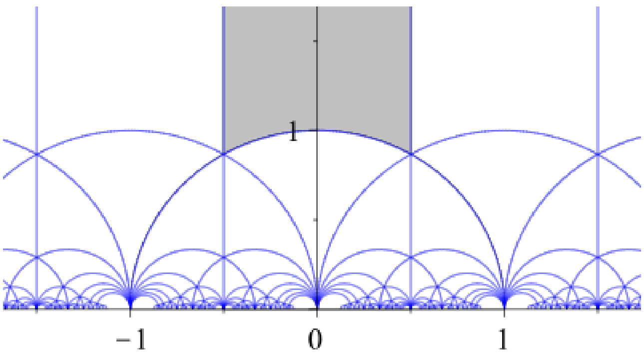


Figure 4: The gray part is the fundamental domain $\mathcal{D}$ of $PSL(2,\mathbb{Z})$. (By Original: Kilom691 Vector: Alexander Hulpke - Own work based on: ModularGroup-FundamentalDomain-01.png, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=59963451)