GW/Hilbert versus AdS/CFT Correspondence

W.Lerche, Heidelberg 6-2023

- Motivation: investigate large distance in AdS/CFT moduli space
- Setting: F1NS5 system for $AdS_3 \times S^3 \times K3$ at $Q_5=1$, topological
- Recap math: Hilbert scheme and reduced, relative GW invariants
- GW/Hilb Correspondence: enumerative counting problem
- Deformation theory = 't Hooft expansion
- Example: all order genus expansion, strong coupling limit
- Caveats!

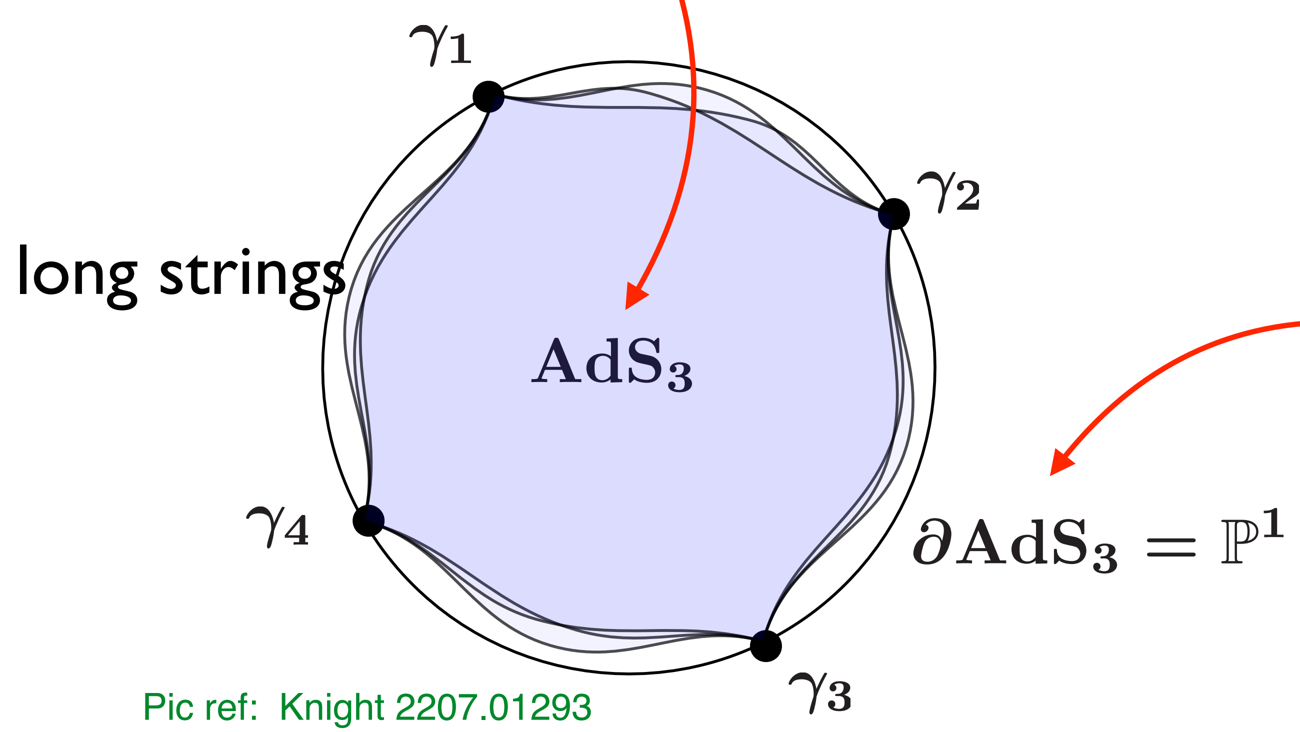
W.L. to appear

Setting: "Topological" AdS_3/CFT_2

- Setting: F1NS5 system at near horizon limit described by $AdS_3 \times S^3 \times K3$. For NS flux $k=Q_5=1$ it is in some sense topological (no continuum of long strings)

Eberhardt,Gaberdiel,Gopakumar,……..

AdS side: free-field world-sheet description ito. $PSU(1,1|2)_1$ WZW model (T_4)



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Boundary CFT at u=0:
Sigma-Model with target space $Sym^d(K3)$

$$d = Q_1 Q_5 + 1 \to \infty$$

u≠0 deforms to $Hilb^d(K3)$

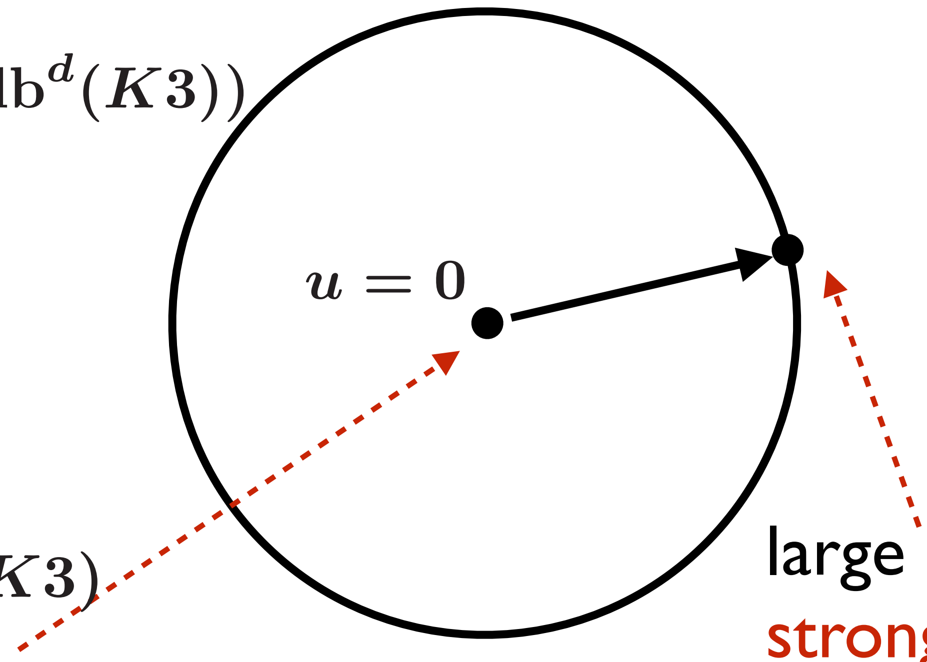
- Comparing correlation functions $\langle \gamma_1 \dots \gamma_n \rangle$ on the CFT side and on the AdS side (mainly for T_4) at large N=d has been a major enterprise over the last 20+ years, with many people involved

Lunin,Mathur;Pakman,Rastelli,Razamat;…. Lima,Sotkov,Staniskov; Li,Troost
Dei,Eberhardt,Knight,Gaberdiel,Gopakumar,……..

Large distance in boundary CFT moduli space

- Sigma-model with target space $\mathrm{Sym}^d(K3)$ resp. $\mathrm{Hilb}^d(K3)$
- Consider deformation of moduli space, in the blow-up modulus u of the orbifold singularity at u=0:

$\mathcal{M}_u(\mathbf{Hilb}^d(K3))$



.. at odds with distance conjecture?

Orbifold point $\mathcal{M}(\mathbf{Sym}^d(K3))$
finite distance sing.
weakly coupled
tensionless strings

large distance singularity,
strongly coupled supergravity regime

Plan: Go beyond conformal perturbation theory by using algebraic geometry.

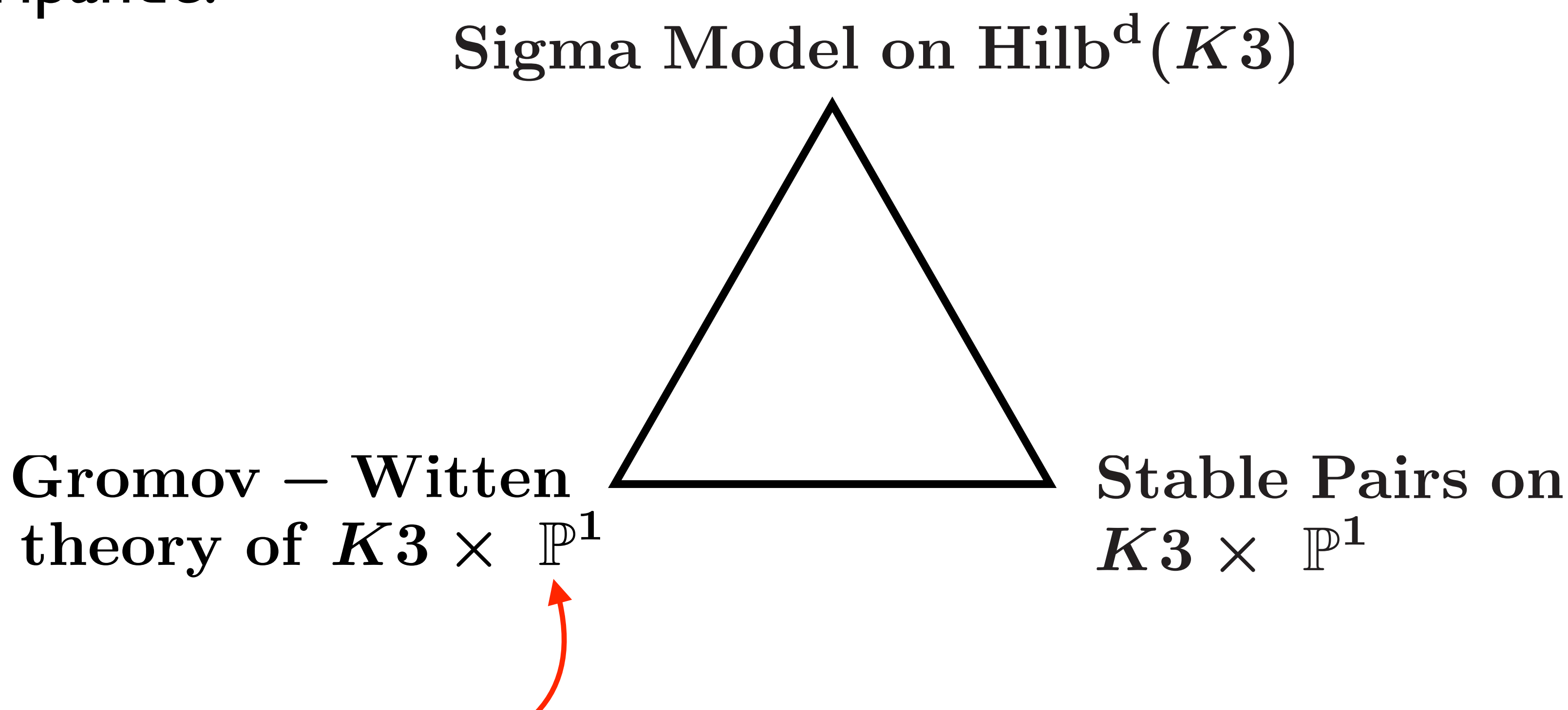
- Benefit: all-order expansion
- Drawback: topological subsector only

Gromow-Witten/Hilbert Correspondence

- So far only few orders in the deformation parameter u has been computed by conformal perturbation theory (hard combinatorics of twist fields).

  Things should be easier in a protected, chiral subsector which allows to use the power of geometrical methods.

  For this we use the GW/Hilbert Correspondence of Oberdieck and Pandharipande: 1406.1139,1411.1514, 1605.05238, 2202.03361



- The idea is to identify $\mathbb{P}^1 = \partial \mathrm{AdS}_3$ and the orbifold blowup mode with the 't Hooft coupling. The counting of coverings maps of degree d is taken care of by GW invariants and leads to an all-genus expansion.

(Co)homology of the Hilbert Scheme Hilb^d^(K3)

- Correlators $\langle \gamma_1 \ldots \gamma_n \rangle$ ... γ_i are cohomology classes of Hilb^d(K3)

- Systematic Fock space construction of (co)homology a la Nakajima:
  Lift cohomology elements while adding cycles of length a to $\alpha \in H^*(K3)$

  $$p_{-a}(\alpha): \; \gamma \in H^*(\mathrm{Hilb}^b(K3), \mathbb{Q}) \longrightarrow p_{-a}(\alpha) \cdot \gamma \in H^*(\mathrm{Hilb}^{a+b}(K3), \mathbb{Q})$$

  (a: windings, spectral flow)

  Thus one can represent write cohomology elements as

  $$\gamma^{(\pi)} = \prod_i p_{-\mu_i}(\alpha_i) 1 \;\in\; H^{q,q}(\mathrm{Hilb}^d(K3), \mathbb{Q})$$

  labelled by "cohomology weighted partitions"

  $$\pi = [(\mu_i, \alpha_i)], \qquad i = 1, \ldots, \ell(\mu)$$

  where μ is a partition of d of length $\ell(\mu)$

  $$\mu = [\mu_1, \ldots \mu_\ell], \quad \ell(\mu) = d - \sum (\mu_i - 1)$$

- Total degree/ R-charge: $q = \sum_{i=1}^{\ell(\pi)} (q_i + (\mu_i - 1))$

Curve classes

- Second homology is one dimension larger than for K3: (21=20+1)

$$H_2(\mathrm{Hilb}^d(K3), \mathbb{Z}) \;=\; H_2(K3, \mathbb{Z}) \,\oplus\, \mathbb{Z}\, A\,, \qquad d > 1$$

Extra class A is exceptional blow-up curve of orbifold singularity of Sym^d(K3), and will be key part of the story

$$A \;=\; p_{-2}(\mathrm{pt}) p_{-1}(\mathrm{pt})^{d-2}\, 1\,, \quad d \geq 2$$

We also consider a curve β_h in K3 with self-intersection $\langle \beta_h, \beta_h \rangle \;=\; 2h-2$ and lift it to Hilb:

$$C(\beta_h) \;=\; p_{-1}(\beta_h)\, p_{-1}(\mathrm{pt})^{d-1}\, 1$$

We pack them together by writing

$$C_{\beta_h,k} \equiv C(\beta_h) + kA \in H_2(\mathrm{Hilb}^d(K3), \mathbb{Z})\,, \qquad k \in \mathbb{Z}$$

Reduced GW Invariants I

- We are interested in Gromow-Witten invariants that count holom. maps from the world-sheet into the target space Hilb^d(K3), relative to the insertions $\gamma_j \equiv \gamma^{(\pi^{(j)})}$ (ie, they pass through the cycles dual to these)

$$\langle \gamma_1, \ldots, \gamma_n \rangle^{\mathrm{Hilb}^d(K3)}_{g;C_{\beta_h,k}} = \int_{[(\overline{\mathcal{M}}_{g,n})(\mathrm{Hilb}^d(K3),C_{\beta_h,k})]^{virt}} ev_1^*(\gamma_1) \cup \ldots ev_n^*(\gamma_k)$$

(g=0 for us)

For these integrals not to vanish, the charge/degree condition must be satisfied:

$$\left(\dim_{\mathbb{C}}(\mathrm{Hilb}^d(K3)) - 3\right)(1-g) \stackrel{!}{=} \sum_{j=1}^{n}(q^{(j)} - 1)$$

- It is well-known that these integrals however do vanish identically for hyperkahler K3 and its associated Hilbert schemes.

  …no world-sheet instantons for N=(4,4) supersymmetric sigma models!

Reduced GW Invariants II

Caveat!

- One way of seeing is to note that deformations of K3 map curve classes of Hodge type (1,1) into (0,2), whose invariants vanish. By deformation invariance, the GW invariants must vanish alltogether.

- We thus need to refine the notion of GW invariants in order to render them non-vanishing. This leads to "reduced" invariants.

Bryan,Leung; Maulik,Pandharipande,Thomas….

One way of achieving this is to consider K3 adiabatically as fiber of CY threefolds. Then the deformations of curves restrict to Hodge Type (1,1). Can also be defined intrinsically to K3, by modifying the obstruction theory.

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{g;C_{\beta_h},k}^{\mathrm{Hilb}^d(K3)} = \int_{[(\overline{\mathcal{M}}_{g,n}(\mathrm{Hilb}^d(K3),C)]^{red}} ev_1^*(\gamma_1) \cup \ldots ev_n^*(\gamma_k)$$

This underlies also the curve counts of the famous Zaslow-Yau and KKV formulas, which apply for d=1 or 0, resp.
(see later).

Gromov-Witten/Hilbert Correspondence

… can now be stated concretely in terms of this key formula:

Oberdieck 1406.1139, 1605.05238, 2202.03361

$$\sum_{k\in\mathbb{Z}} y^k \langle\, \gamma_1, \ldots, \gamma_n \,\rangle^{\mathrm{Hilb}^d(K3)}_{0;C_{\beta_h,k}} = u^{2d-\Delta(\pi)} \sum_{g\geq 0} u^{2g-2} \langle\, \gamma_1, \ldots, \gamma_n \,\rangle^{K3\times\mathbb{P}^1/\{z_1,\ldots,z_n\}}_{g;(\beta_h,d)}$$

with $\quad y \;=\; e^{2\pi i(z+1/2)} \;\equiv\; -e^{-u}$

$$\Delta(\pi) \equiv \sum_{j=1}^{n} \sum_{i=1}^{\ell(\pi^{(j)})} (\mu_i^{(j)} - 1)$$

LHS = Hilbert-side:
Exponential DT-like expansion

Insertions $\gamma_j \equiv \gamma^{(\pi^{(j)})}$ labelled by terms of cohomology weighted partitions, as discussed before

RHS = GW-side:
Power-like genus expansion, counting holomorphic maps of genus g curves into branched coverings of P¹xK3 of degree d

What is meant by $\langle\, \gamma_1, \ldots, \gamma_n \,\rangle^{K3\times\mathbb{P}^1}_{g;(\beta_h,d)}$?

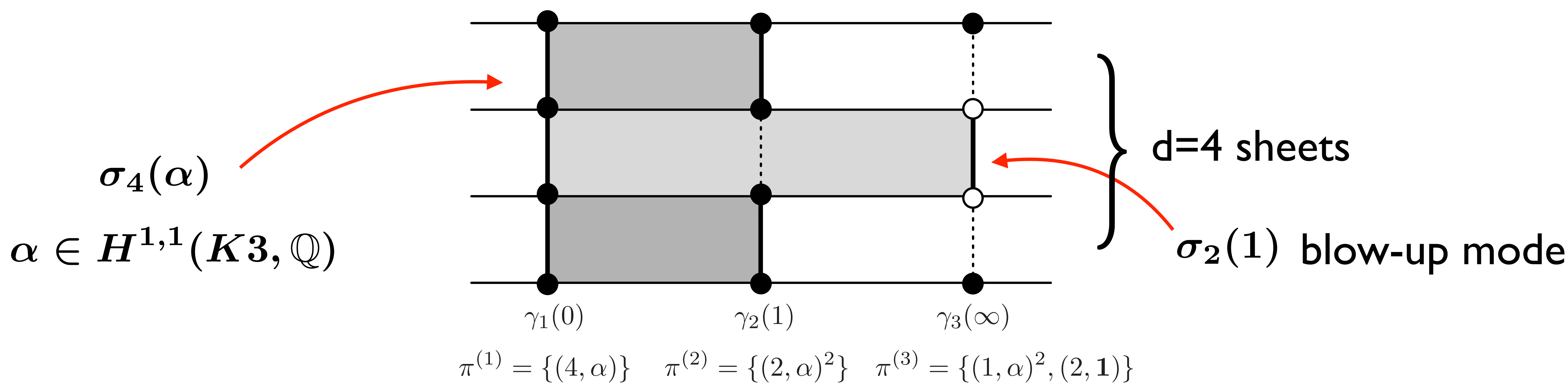
Ramified coverings of P^1 of degree d

- On the RHS the insertions in $\langle \gamma_1, \ldots, \gamma_n \rangle^{K3\times\mathbb{P}^1}_{g;(\beta_h,d)}$ have an equivalent meaning, and are labelled by the same cohomology weighted partitions

$$\pi^{(j)} \;=\; [(\mu_i^{(j)}, \alpha_i^{(j)})]\,, \qquad i = 1, \ldots, \ell(\mu^{(j)})$$

They are combinations of twist fields: $\gamma_j \equiv \gamma^{(\pi^{(j)})} \;=\; \prod_{i=1}^{\ell(\mu)} \sigma_{\mu_i}(\alpha_i)$

The insertions of those thus lift P^1 to (in general) higher genus covering curves, eg:



This example corresponds to a genus 0 covering curve with 3 massive and 3 massless insertions

GW invariants count covering maps

- In this way the invariant $\langle \gamma_1, \ldots, \gamma_n \rangle^{K3\times\mathbb{P}^1}_{g;(\beta_h,d)}$ counts degree d maps $\Sigma_{g,n} \to K3 \times \mathbb{P}^1/\{z_1, \ldots, z_n\}$

  This is closely related to GW/Hurwitz correspondence, which allows to count such branched coverings in terms of topological gravity.

  Okounkov,Pandharipande math/0204305

  Essentially one maps twist fields to gravitational descendants: $\sigma_{k+1} \leftrightarrow \tau_k$

  and thus $\gamma^{(\pi^{(j)})} \longleftrightarrow M \cdot \prod_{i=1}^{\ell(\mu^{(j)})} \tau_{\mu_i^{(j)}-1}(\alpha_i^{(j)}))$

  "completed cycles"

  The advantage is that computations in topological gravity may be simpler as one can bypass the complicated combinatorics of twist fields.

- The basic gravitational descendant τ_1 corresponds to the universal twist field σ_2 which figures as the marginal operator that deforms the symmetric orbifold away from the orbifold point: $\delta S \sim u \int G^- \bar{G}^- \sigma_2$

Generating function

- Pack everything together and also sum over $q \equiv e^{2\pi i \tau}$:

$$\mathcal{F}^{(d)}_{\{\pi^{(j)}\}}(q,u) \;=\; u^{2d-\Delta(\pi)} \sum_{g,h\geq 0} u^{2g-2} q^{h-1}\, \mathrm{GW}^{(d)}_{g,h,\pi}$$

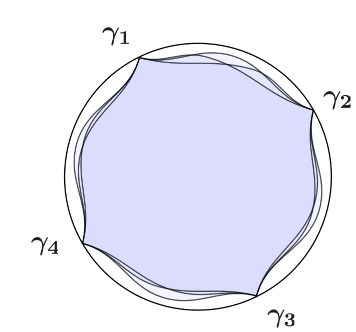
$$\mathrm{GW}^{(d)}_{g,h,\pi} \;=\; \left\langle\, c(\Lambda^\vee)\,;\, \prod_{j}^{n} \gamma^{(\pi^{(j)})}(z_j) \,\middle|\, \tau_1^{N_\tau}(\omega) \,\right\rangle^{K3\times\mathbb{P}^1/\{z_1,\ldots,z_n\}}_{g;(\beta_h,d)}$$

Hodge classes

Degree rule picks out correct number N_τ of τ_1 insertions:

$$\left(\dim_{\mathbb{C}}(\mathbb{P}^1)-3\right)(1-g) + \int_\beta c_1(\mathbb{P}^1) \;\equiv\; 2g-2+2d \;\stackrel{!}{=}\; \Delta(\pi)+N_\tau$$

- Consider $\mathcal{F}^{(d)}_{\{\pi^{(j)}\}}(q,u)$ as a generating function for boundary conditions specified by the insertions $\gamma_j \equiv \gamma^{(\pi^{(j)})}$,
perturbed by the K3 modulus τ and deformation modulus u



’t Hooft Expansion

- How does the genus expansion in u relate to expansion in the string coupling on the AdS side?

 They can’t be the same as there is a non-zero string coupling even at u=0.

 The point is that the ramification profiles $\{\pi^{(j)}\}$ define a “bare” curve $\Sigma^{(0)}$ of genus $g_0 = 1 - d + \frac{1}{2}\Delta(\pi)$ and we perturb this background with the orbifold blow-up modulus u. This adds more and more handles to $\Sigma^{(0)}$.

- For each bare geomtry, weigh the generating function by the string coupling constant:

$$g_s{}^{2g_0-2}\mathcal{F}^{(d)}_{\{\pi^{(j)}\}}(q,u) = \sum_{g\geq g_0, h\geq 0} g_s{}^{2g_0-2}\, u^{2g-2g_0}\, q^{h-1}\, \mathrm{GW}^{(d)}_{g,h,d,\pi}$$

$$= \sum_{g\geq g_0, h\geq 0} N^{1-g}\, \lambda^{2g-2g_0}\, q^{h-1}\, \mathrm{GW}^{(d)}_{g,h,N,\pi}$$

 Here we have set $d = N, \quad g_s = \frac{1}{\sqrt{N}}, \quad u = \frac{\lambda}{\sqrt{N}}$

Thus the orbifold deformation parameter u maps to the ’t Hooft parameter λ

Example

- Most basic but significant bare geometry corresponds to completely unramified covering curve:

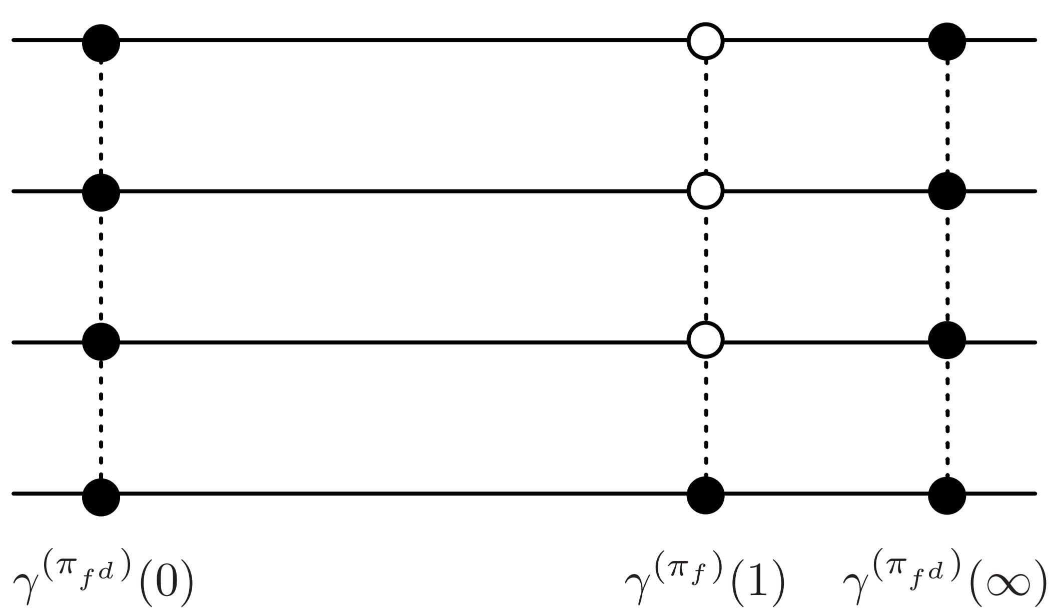
$$\gamma^{(\pi_{f^d})} = p_{-1}(f)^d \, 1 \in H^{d,d}(\mathrm{Hilb}^d(K3), \mathbb{Q})$$

$$\gamma^{(\pi_f)} = p_{-1}(f) p_{-1}(1)^{d-1} \, 1 \in H^{1,1}(\mathrm{Hilb}^d(K3), \mathbb{Q})$$

$$\beta_h = b + hf \in H^{1,1}(K3, \mathbb{Q})$$

(f = ell fiber class)

It is highly disconnected (g_0=1-d):



world-sheets of tensionless strings

2d+1
massless Kähler moduli

Generating function

- This leads to the generating function:

$$\mathcal{F}^{(d)}_{\{\pi_{f^d},\pi_f,\pi_{f^d}\}}(q,u) \;=\; \sum_{\substack{k\in\mathbb{Z}\\ h\geq 0}} \langle\, \gamma^{(\pi_{f^d})}\,\gamma^{(\pi_f)}\,\gamma^{(\pi_{f^d})}\rangle^{\mathrm{Hilb}^d(K3)}_{0;C_{\beta_h,k}}\, y^k q^{h-1}$$

… which can be solved in closed form:

1406.1139,1411.1514,
1605.05238, 2202.03361

$$\mathcal{F}^{(d)}_{\{\pi_{f^d},\pi_f,\pi_{f^d}\}}(q,u) \;=\; \frac{1}{(d!)^2}\,\varphi_{-2,1}(\tau,z)^{d-1}\,\frac{1}{\Delta(\tau)}$$

- Note:

  d=1: Yau-Zaslow formula

  d=0: KKV formula, appears in black hole partition functions, ell genera

Jacobi Form

- $$\varphi_{-2,1}(\tau,z) = \frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^6} = u^2 + \mathcal{O}(u^4)\,, \qquad q = e^{2\pi i\tau},\quad u \equiv -2\pi i z$$

  = standard Jacobi generator with weight w=-2 and index m=1:

$$\varphi_{w,m}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right) = (c\tau+d)^w e^{2\pi i\frac{mc}{c\tau+d}z^2}\varphi_{w,m}(\tau,z) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$$\varphi_{w,m}(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m(\lambda^2\tau+2\lambda z)}\varphi_{w,m}(\tau,z)\,, \quad \lambda,\mu\in\mathbb{Z}$$

- At infinite K3 fiber volume, q=0: $\varphi_{-2,1}(0,u) = \mathcal{S}(u)^2$

  where $\mathcal{S}(u) = 2\sinh\left(\frac{u}{2}\right) = -i(y^{1/2}+y^{-1/2})$

- Integral Gopakumar-Vafa expansion: $\mathrm{GV}^{(d)}_{g,h} \in \mathbb{Z}$

$$\mathcal{F}^{(d)}_{\{\pi_{fd},\pi_f,\pi_{fd}\}}(q,u) = \frac{1}{(d!)^2\Delta(q)}\sum_{g,h\geq 0}\mathrm{GV}^{(d)}_{g,h}\, q^h\, \mathcal{S}(u)^{2g-2+2d}$$

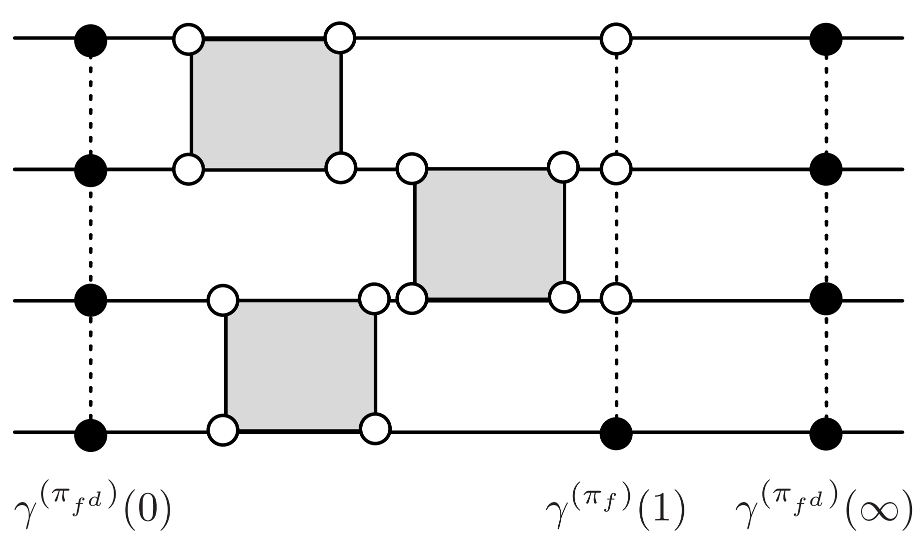
Genus expansion around orbifold point u=0

- Switching on u starts linking the d disconnected world-sheets, which may be viewed as a binding process of the tensionless strings

  After 2d-2 insertions of the twist field, a connected g=0 world-sheet emerges, which can be viewed as a d times wrapped long string.

- At q=0 one expects an expansion into simple Hurwitz numbers,

$$H_d(u) \;=\; u^{2d-2}\sum_{g\geq 0} u^{2g}\,\frac{1}{(2g+2d-2)!}\,H_{g,d}^{(1^d,1^d)}\,,$$



but actually it does not: $\mathcal{F}^{(d)}(u) \;\sim \mathcal{S}(u)^{2d-2} \;\not\sim\; H_d(u)\,,\quad d\neq 1,2$ eg. d=3:

$$\mathcal{F}^{(3)}(u) \;=\; \frac{u^4}{4!}\,6\,H_{0,3}+\frac{u^6}{6!}(6H_{1,3}-H_{0,4})+\frac{u^8}{8!}(6H_{2,3}-H_{1,4}+\frac{9}{20}H_{0,5})+\ldots$$

Strong coupling expansion u→∞

- More interesting but least accessible is the large distance limit, $y = -e^{-u} \to 0$

Generating function is almost double periodic, (spectral flow, theta-shifts) ie.,

$$\mathcal{F}^{(d)}(\tau, z + \lambda\tau + \mu) \;\propto\; q^{-\lambda^2(d-1)}(-y)^{-2\lambda(d-1)}\mathcal{F}^{(d)}(\tau, z)$$

….but not quite due to to the non-vanishing Jacobi index, m=d-1.

This expresses non-convergence outside the fundamental domain

$$F_z \;=\; \{\, z\,|\, 0 \leq \mathrm{Im} z \;<\; \mathrm{Im}\tau\}$$

Leading term as y→0 is:

$$\mathcal{F}^{(d)}(q,y) \;\propto\; y^{1-d}q^{-1}\sum \mathrm{GV}^{(d)}_{g,g}\left(\frac{q}{y}\right)^{g} \;\propto\; y^{1-d}q^{-1}\left(1+\frac{q}{y}\right)^{2d-2}$$

- Singularity does not seem repairable,
probably non-perturbative completion must take over.

Matrix model?

Summary

- Aim: understand global moduli space of the orbifold blow-up mode u
- Framework: topological toy model of boundary of AdS_3 x S^3 x K3
- Based on Gromow-Witten/Hilbert correspondence that maps to an all-genus expansion. Up to rescaling, u figures as a 't Hooft parameter.
- Basic example is exactly solvable in terms of a Jacobi form; however strong coupling region remains out of control
- Caveats: requires to consider reduced GW invariants, whose physical significance remains unclear. Wall crossing, stability?
- Outlook: generalize to more general base geometries, find resurgent, non-perturbative completion in terms of matrix model