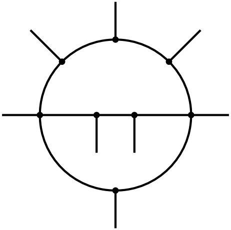
Motivic geometry of two-loop Feynman integrals



Andrew Harder
Lehigh University

Joint with Chuck Doran and Pierre Vanhove

Feynman graphs

Suppose Γ is a graph, $D \in 2\mathbb{Z}_{\geq 0}$ is space-time dimension.

1. $E_\Gamma, V_\Gamma, H_\Gamma$, edges, vertices and half (external) edges
2. To each half-edge, we associate a momentum $p_h \in \mathbb{R}^{1,D-1}$, satisfying;

$$\sum_{h \in H_\Gamma} p_h = 0 \qquad \text{(conservation of momentum)}$$

3. To each internal edge we associate a mass $m_e \in \mathbb{R}_{\geq 0}$.

Simplifications:

- View masses and momenta as complex numbers.
- No 1-valent vertices.
- For each vertex attach a single half-edge (write p_v instead of p_h).

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Symanzik polynomials I

Consider the ring $\mathbb{C}[x_e \mid e \in E_\Gamma]$.

$\mathbf{U}_\Gamma$ polynomial

Suppose T is a spanning tree of Γ,

$$x^T = \prod_{e \notin T} x_e, \qquad \mathbf{U}_\Gamma = \sum_{\text{spanning trees}} x^T.$$

$\mathbf{V}_\Gamma$ polynomial

Suppose $T_1 \cup T_2$ is a spanning 2-tree of Γ.

$$x^{T_1 \cup T_2} = \prod_{e \notin T_1 \cup T_2} x_e, \quad \mathbf{V}_\Gamma = \sum_{\text{spanning 2-trees}} s_{T_1 \cup T_2} x^{T_1 \cup T_2}$$

Here $s_{T_1 \cup T_2} = \left(\sum_{v \in T_1} p_v\right)^2 = \left(\sum_{v \in T_2} p_v\right)^2$.

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Symanzik polynomials II

$\mathbf{F}_\Gamma$ polynomial

$$\mathbf{F}_\Gamma = \mathbf{U}_\Gamma \left(\sum_e m_e^2 x_e \right) + \mathbf{V}_\Gamma .$$

$$\deg \mathbf{U}_\Gamma = \ell(\Gamma), \qquad \deg \mathbf{F}_\Gamma = \ell(\Gamma) + 1$$

Example:

$\mathbf{U}_\Gamma = x_0 x_1 + x_1 x_2 + x_0 x_2$
$\mathbf{V}_\Gamma = x_0 x_1 x_2$
$\mathbf{F}_\Gamma = \mathbf{U}_\Gamma (m_0^2 x_0 + m_1^2 x_1 + m_2^2 x_2) + q^2 \mathbf{V}_\Gamma$

Remark: In nearly *all* examples, $V(\mathbf{F}_\Gamma)$ is *singular*.

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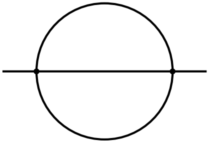

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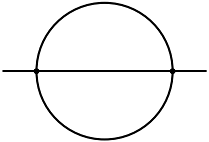

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Feynman integrals

$$\boxed{I_\Gamma(m,s) = \int_\sigma \frac{\mathbf{U}_\Gamma^{e_\Gamma - D(\ell+1)/2}}{\mathbf{F}_\Gamma^{e_\Gamma - D\ell/2}} \Omega_0}$$

$$\sigma = \{[x_1 : \cdots : x_{e_\Gamma}] \in \mathbb{P}^{e_\Gamma - 1} \mid x_i \in \mathbb{R}_{\geq 0}\}$$

$$\Omega_0 = \sum_{i=1}^{e_\Gamma} (-1)^i x_i \left(dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{e_\Gamma} \right)$$

Periods and mixed Hodge structures

$$\left[\frac{\mathbf{U}_\Gamma^{e_\Gamma - D(\ell+1)/2}}{\mathbf{F}_\Gamma^{e_\Gamma - D\ell/2}}\Omega_0\right] \in \mathrm{H}_{\mathrm{dR}}^{e_\Gamma - 1}(\mathbb{P}^{e_\Gamma - 1} - Z_{\Gamma,D}, B_\Gamma)$$

where $\quad Z_{\Gamma,D} = V(\mathbf{F}_\Gamma), V(\mathbf{F}_\Gamma \mathbf{U}_\Gamma),$ or $V(\mathbf{U}_\Gamma),$ and $B_\Gamma = V\left(\prod_{i=0}^{e_\Gamma} x_i\right)$

However

$$[\sigma] \notin \mathrm{H}_{e_\Gamma - 1}(\mathbb{P}^{e_\Gamma - 1} - Z_{\Gamma,D}, B_\Gamma) \quad \text{since} \quad \sigma \cap Z_{\Gamma,D} \neq \emptyset$$

Theorem: (Bloch–Esnault–Kreimer, Brown) After appropriate (toric) blow up $b : \mathbb{P}_\Gamma \to \mathbb{P}^{e_\Gamma - 1}$ and modification of σ, the Feynman integral is a well-defined relative period.

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The mixed Hodge structure on $\mathrm{H}^{e_\Gamma - 1}(\mathbb{P}_\Gamma - Z_\Gamma, B_\Gamma)$ is (up to mixed Tate factors) an iterated extension of mixed Hodge structures of the shape

$$\mathrm{H}^*(\mathbb{P}_{\Gamma-\Gamma'} - Z_{\Gamma-\Gamma',D}; \mathbb{Q}), \quad \mathrm{H}^{e_{\Gamma//\Gamma'}-1}(\mathbb{P}_{\Gamma//\Gamma'} - Z_{\Gamma//\Gamma'}; \mathbb{Q})$$

Goal. We want to study the mixed Hodge structures on the cohomology groups of $Z_{\Gamma,D}$.

(I.e., we do not study Feynman integrals individually, rather we study a space of functions in which they live).

From here on, will study the case where $Z_{\Gamma,D} = V(\mathbf{F}_\Gamma)$.

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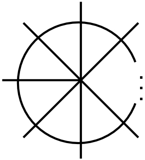
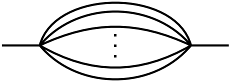


Sunset family



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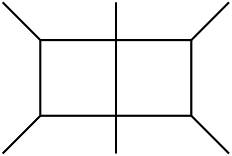

Other results:

- (Belkale–Brosnan) The motives of $\mathbf{U}_\Gamma$ are in some sense general.
- (Doryn) Zig-zag family of graphs.
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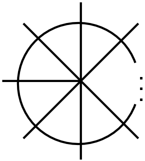

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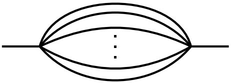

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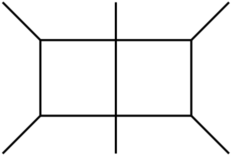

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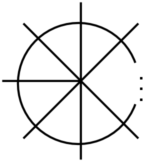

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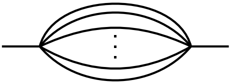

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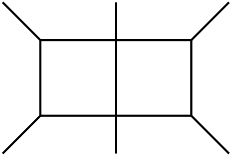

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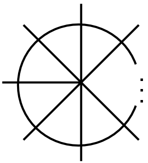

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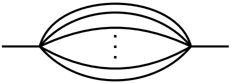

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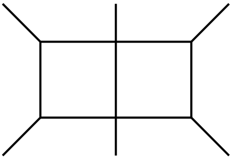

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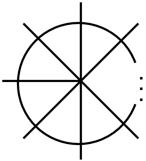

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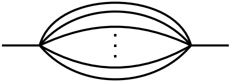

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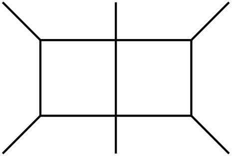

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Lemma (Doran-H.-Vanhove). If Γ is a Feynman graph, $\ell(\Gamma) > 1$, and $e_1, \ldots, e_k$ form a chain of edges, then $V(\mathbf{F}_\Gamma)$ admits a quadric fibration over $\mathbb{P}^{e_\Gamma - k - 1}$.

"Proposition". Suppose $\pi : Z \to \mathbb{P}^n$ is a quadric fibration of dimension n with generic corank r and let $\mathcal{D}$ be a divisor along which the corank of quadrics increases.

(1) If $r \neq 0$ then $\mathrm{H}^{\mathrm{middle}}(Z, \mathbb{Q})$ is mixed Tate.

(2) If $r = 0$ the relative dimension of π is odd, then the cohomology of $\mathrm{H}^{\mathrm{middle}}(Z, \mathbb{Q})$ is determined by a double cover of B ramified along $\mathcal{D}$.

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Chains of edges and quadric fibrations

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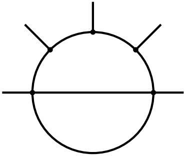
Example: $\Gamma_{a,1,1}$ graphs



$\|$

$\Gamma_{a,1,1}$

$Z_{\Gamma_{a,1,1},D}$ admits a quadric fibration over $\mathbb{P}^1$.

Possible cases:

1. Dimension large then *all* fibres are singular. Thus MHS is mixed Tate.
2. Dimension small and a is even, then MHS is mixed Tate.
3. Dimension small and a is odd then five singular fibres; four are nodal, one has rank 1. Thus MHS is a mixed Tate extension of $\mathrm{H}^1(E;\mathbb{Q})$.

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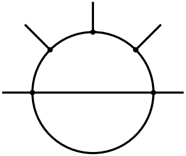

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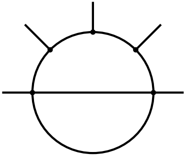

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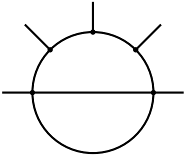

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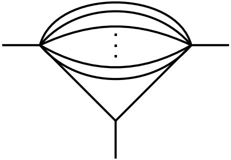

$V(\mathbf{F}_\Gamma)$ admits a conic fibration over $\mathbb{P}^{n-1}$.

The discriminant locus is a union of two distinct sunset Calabi–Yau $(n-1)$-folds.

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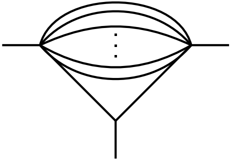

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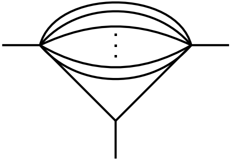

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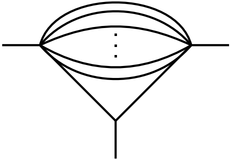

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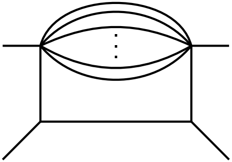

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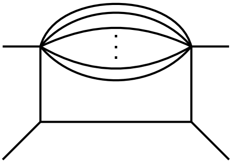

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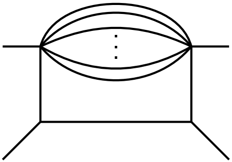

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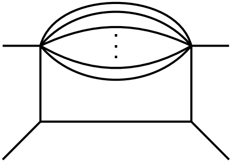

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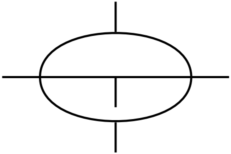
This is a family of cubic fourfolds in $\mathbb{P}^5$.



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The discriminant locus is a nodal quartic surface along with a hyperplane.

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Computations suggest that the generic Picard rank of this K3 surface is 11.

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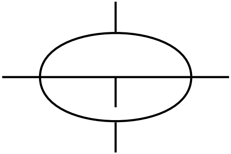

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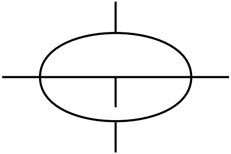

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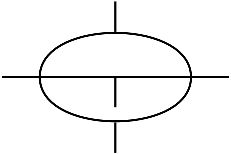

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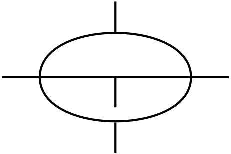

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Feynman cubics ($\ell = 2$)

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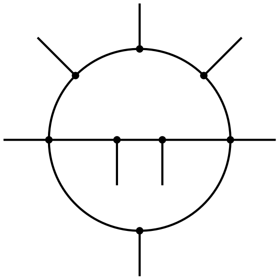

$\deg \mathbf{F}_{\Gamma_{a,b,c}} = 3$
$\deg \mathbf{U}_{\Gamma_{a,b,c}} = 2$

The MHS attached to $\mathbf{U}_{\Gamma_{a,b,c}}$ is mixed Tate ("easy") so we will try to understand the case where $Z_{\Gamma_{a,b,c},D} = V(\mathbf{F}_{\Gamma_{a,b,c}})$.

Physics computations suggest that these motives are simple – that is, they are mixed Tate, or that they come from elliptic curves, at least when one of a, b, or $c = 1$.

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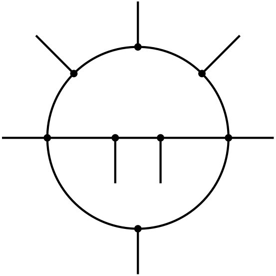

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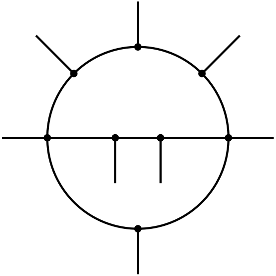
$\Gamma_{a,b,c} \quad =$ 

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The main result (planar two-loop graphs)

Theorem (Doran-H.-Vanhove). Suppose that $b = 0, 1$ (that is, $\Gamma_{a,b,c}$ is a planar graph). Then if $\mathrm{H}^*(V(\mathbf{F}_{\Gamma_{a,b,c}}); \mathbb{Q})$ is contained in

$$\mathbf{MHS}_{\mathbb{Q}}^{\mathrm{hyp}} = \text{Extension-closed subcategory of } \mathbf{MHS}_{\mathbb{Q}} \text{ generated by cohomology of hyperelliptic curves.}$$

If a or $c \leq 2$ then we can replace "hyperelliptic" with "elliptic". If dimension is large enough relative to a and c then cohomology is mixed Tate.

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Proof sketch

Proposition. If a cubic hypersurface X contains a linear subspace L of codimension 1, then $\mathrm{H}^*(X;\mathbb{Q}) \in \mathbf{MHS}_{\mathbb{Q}}^{\mathrm{hyp}}$.

Proof. Take the pencil of hyperplane sections in X containing L. This induces a quadric fibration over $\mathbb{P}^1$. The cohomology of a quadric fibration over $\mathbb{P}^1$ is either (a) mixed Tate (if relative dimension is even) or hyperelliptic (if relative dimension is odd). □

Observation: The cubic equation $\mathbf{F}_{\Gamma_{a,0,c}}$ vanishes along a linear subspace of codimension 1.

However, the cubic $\mathbf{F}_{\Gamma_{a,1,c}}$ does *not* vanish along a linear subspace of codimension 1!

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Proof sketch (continued)

There is a birational map: $\phi : V(\mathbf{F}_{\Gamma_{a,1,c}}) \dashrightarrow X_{a,1,c}$ and open subsets

$$\begin{array}{ccc} U & \longrightarrow & W \\ \downarrow & & \downarrow \\ V(\mathbf{F}_{\Gamma_{a,1,c}}) & \overset{\phi}{\dashrightarrow} & X_{a,1,c} \end{array}$$

so that

(1) $X_{a,1,c}$ is a cubic containing a codimension 1 linear subspace,

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Motivation coming from Picard–Fuchs operators

Lairez–Vanhove: For many 2-loop Feynman graphs, compute the Picard–Fuchs operators of the family of hypersurfaces

$$\mathbf{F}_\Gamma(t) = \mathbf{U}_\Gamma \left(\sum_{e \in E_\Gamma} m_e^2 x_e \right) + t\mathbf{V}_\Gamma, \qquad \omega_\Gamma = \frac{\mathbf{U}_\Gamma^{e_\Gamma - \mathrm{D}(\ell+1)/2}}{\mathbf{F}_\Gamma^{e_\Gamma - \mathrm{D}\ell/2}} \Omega$$

I.e., find a differential operator $\mathcal{L}_\Gamma \in \mathbb{C}[t, \partial_t]$ so that

$$\mathcal{L}_\Gamma[\omega_\Gamma] = [0] \in \mathrm{H}_{\mathrm{dR}}^{e_\Gamma - 1}(\mathbb{P}^{e_\Gamma - 1} - V(\mathbf{F}_\Gamma))$$

Remark: $I_\Gamma(t)$ satisfies an inhomogeneous ODE

$$\mathcal{L}_\Gamma I_\Gamma(t) = f(t)$$

where $f(t)$ is a sum of integrals associated to $\Gamma - e$, and $\Gamma // e$.

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Structure theorem for Lairez–Vanhove ODEs

Observation: Frequently, $\mathcal{L}_\Gamma$ factors into components which are

- Picard–Fuchs operators of families of elliptic curves,
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Sol($\mathcal{L}_\Gamma$) is a subquotient of $\mathcal{H}_\Gamma = \mathbb{C}$-VMHS attached to $V(\mathbf{F}_\Gamma(t))$.

Weight filtration on $\mathcal{H}_\Gamma \rightsquigarrow$ Factorization of $\mathcal{L}_\Gamma$

Weight graded pieces of $\mathcal{H}_\Gamma \rightsquigarrow$ Factors of $\mathcal{L}_\Gamma$

Theorem (Doran–H.–Vanhove). If Γ is an $(a, 1, c)$-type graph then $\mathcal{L}_\Gamma$ factors as:

- Factors with finite monodromy ($\sim$ Liovillian).
- Picard–Fuchs operators for families of hyperelliptic curves.

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Theorem (Doran–H.–Vanhove). If Γ is an $(a, 1, c)$-type graph then $\mathcal{L}_\Gamma$ factors as:

- Factors with finite monodromy ($\sim$ Liovillian).
- Picard–Fuchs operators for families of hyperelliptic curves.

Structure theorem for Lairez–Vanhove ODEs

Observation: Frequently, $\mathcal{L}_\Gamma$ factors into components which are

- Picard–Fuchs operators of families of elliptic curves,
- Liouvillian ODEs ($=$ Solvable differential Galois group).

Sol$(\mathcal{L}_\Gamma)$ is a subquotient of $\mathcal{H}_\Gamma = \mathbb{C}$-VMHS attached to $V(\mathbf{F}_\Gamma(t))$.

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Remarks

(1) If $D = 4$ then *only* elliptic curves show up, for all a, c.

- If dimension is high enough, relative to a, c, the quadric fibrations only contribute mixed Tate components.
- Analyze the remaining cases and conclude that only elliptic curves appear.

(2) If $D = 6$ we start to see curves of higher genus appearing. E.g., double box:

- If $D \leq 2$ then only rational curves appear.
- If $D = 4$ then a family of elliptic curves appear (c.f. Bloch).
- If $D = 6$ then a family of genus 2 curves appear.

(3) For arbitrary (a, b, c) we expect that the same proof will apply to give a bound on the Hodge structures appearing; e.g., if $\min\{a, b, c\} = 2$, the Hodge structure on $V(\mathbf{F}_{\Gamma_{a,b,c}})$ comes from surfaces etc..

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