Tameness in Hodge Theory and Physics

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Based on:

2302.04275 Part II
2210.10057 Part I — with Michael Douglas, Lorenz Schlechter

2112.08383 — TG

2112.06995 — with Ben Bakker, Christian Schnell, Jacob Tsimerman

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Motivation

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- Tameness in Hodge theory: (biased, incomplete list)
  - Tameness of the period map, mixed period map, period integrals [Bakker, Klingler Tsimerman '18]…
  - Ax-Schanuel conjecture for Hodge structures [Bakker,Tsimerman '17]…
    several subsequent generalizations, e.g. to mixed Hodge structures
  - Finiteness of self-dual integral classes [Bakker,TG,Schnell,Tsimerman '21]
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  - 12d manifold: $\mathbb{S} \times Y$ ← Calabi-Yau fourfold
  - 4-form: $G_4 \in H^4(Y, \mathbb{Z})$ $\int_Y G_4 \wedge G_4 = \ell$ $*G_4 = G_4$ (in cohom.)

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- Conjecture [Douglas ’03] [Acharya,Douglas ’06]:

  Number of distinct solutions of string theory with bounds on vacuum energy, KK scale, compactification volume are finite

  $\rightarrow$ in the above setting: finitely many choices for G_4.

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Theorem [Bakker,TG,Schnell,Tsimerman]: For integer $\ell > 0$, the locus of integral self-dual classes

$$\mathcal{S}_\ell = \Big\{ (x,v) \in E :\ v \in H_{\mathbb{Z},x} \text{ and } C_x v = v \text{ and } Q(v,v) = \ell \Big\}$$

is a set definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$

$\to$ $\mathcal{S}_\ell$ has finitely many connected components

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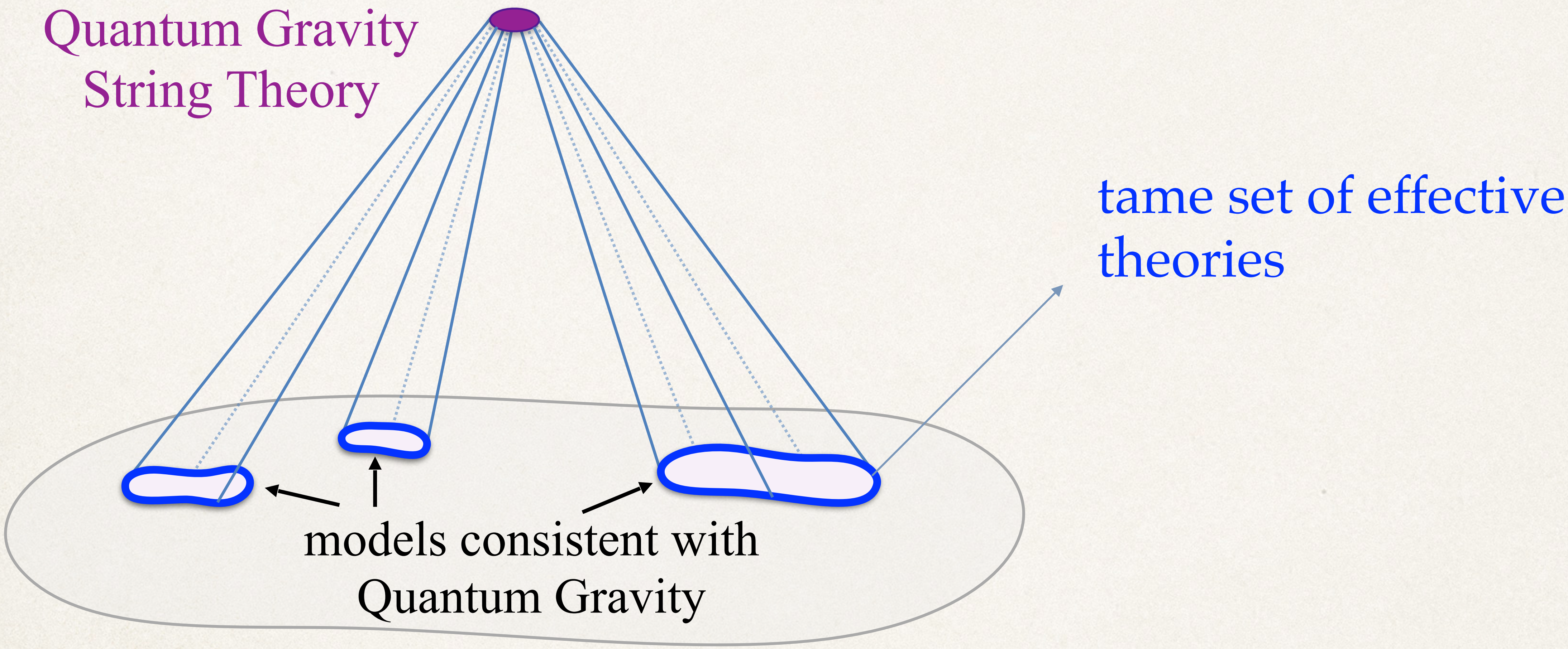

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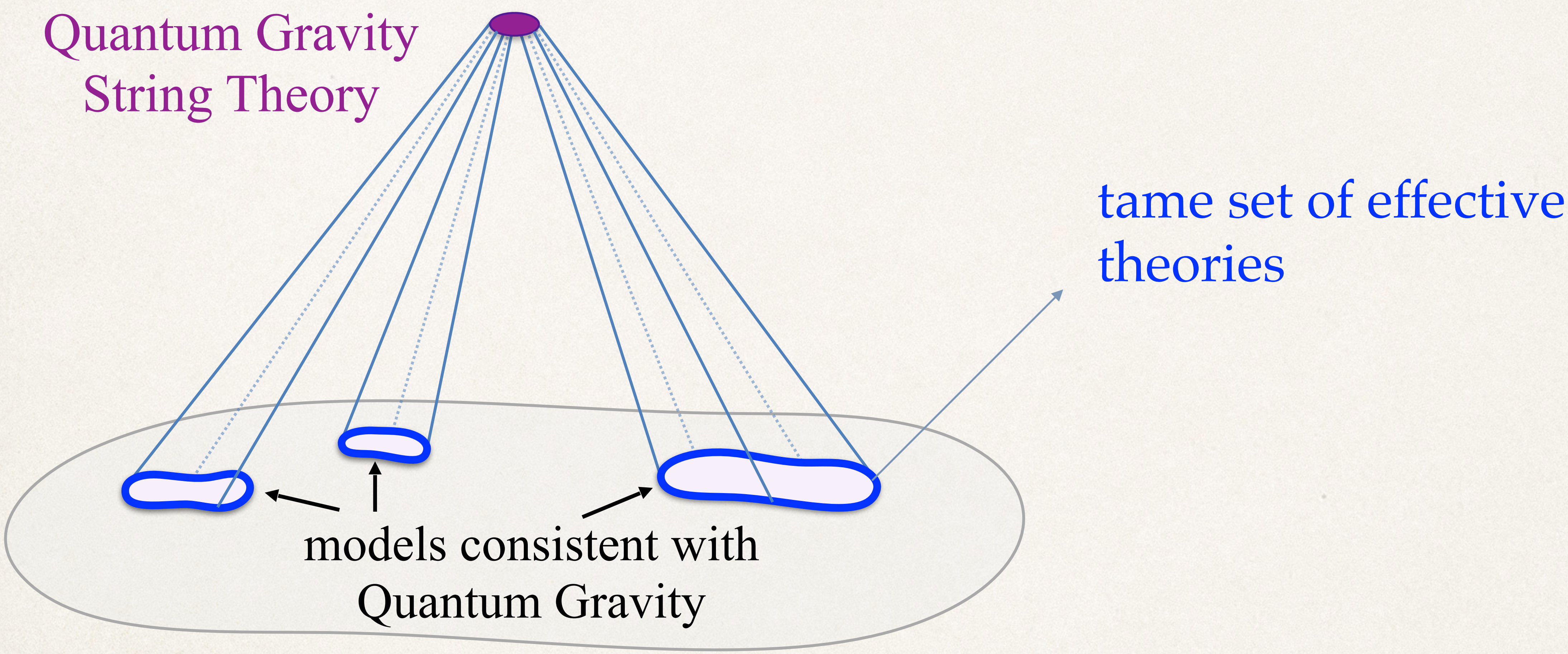

- Tameness in Quantum Field Theories (QFTs) [Douglas,TG,Schlechter '22+'23]
  → use of many of the recent results on tameness in Hodge theory
  → several new conjectures

Tameness and o-minimal structures

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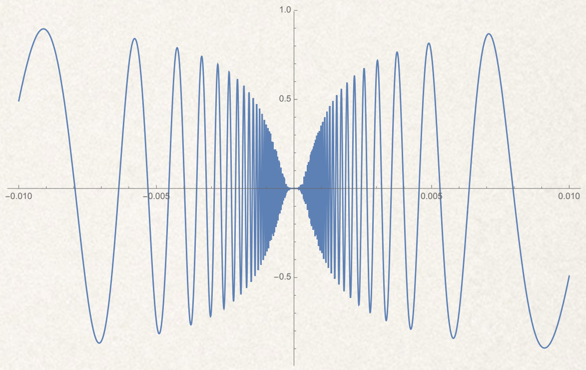

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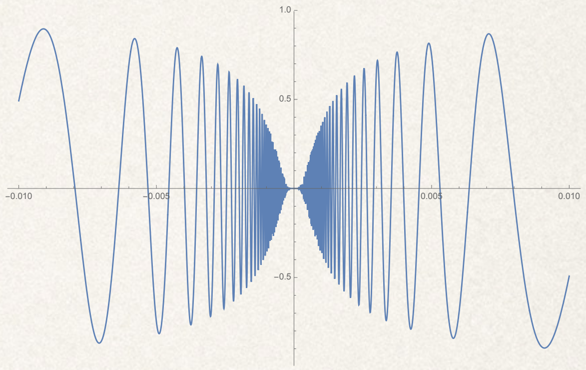

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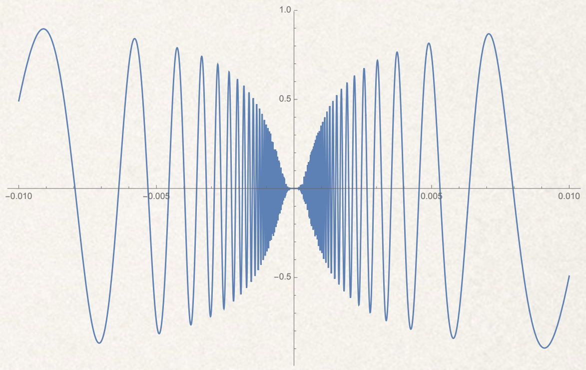

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- Grothendieck's dream:
  develop mathematical framework for geometry
  → tame topology [Esquisse d'un programme]

Tameness - Definition

- structure $\mathcal{S}$: collect subsets of $\mathbb{R}^n$, $n = 1, 2, \ldots$
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  - closed under projections (existential quantifier $\exists$)
  - sets defined by all real polynomials included (algebraic sets)

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sets in o-minimal structure $\mathcal{S}$: tame sets

functions with graph being a tame set: tame functions

$\rightarrow$ tame manifold, tame bundles… a tame geometry

Examples of o-minimal structures

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- Logic perspective: $\mathbb{R}_{\mathcal{F}} = \langle \mathbb{R}; +, \cdot, -, >, \mathcal{F} \rangle$ $\mathcal{F} = \{f_1, f_2, \ldots\}$

all formulas using these symbols and ∧, ∨, ¬, ∃, ∀

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    F_i functions in o-minimal structure $\tilde{\mathbb{R}}$ [Speissegger '97]

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    ⋮
  - structure including $\Gamma(x)|_{(0,\infty)}$ and $\zeta(x)|_{(1,\infty)}$ [Rolin,Servi,Speissegger '22]

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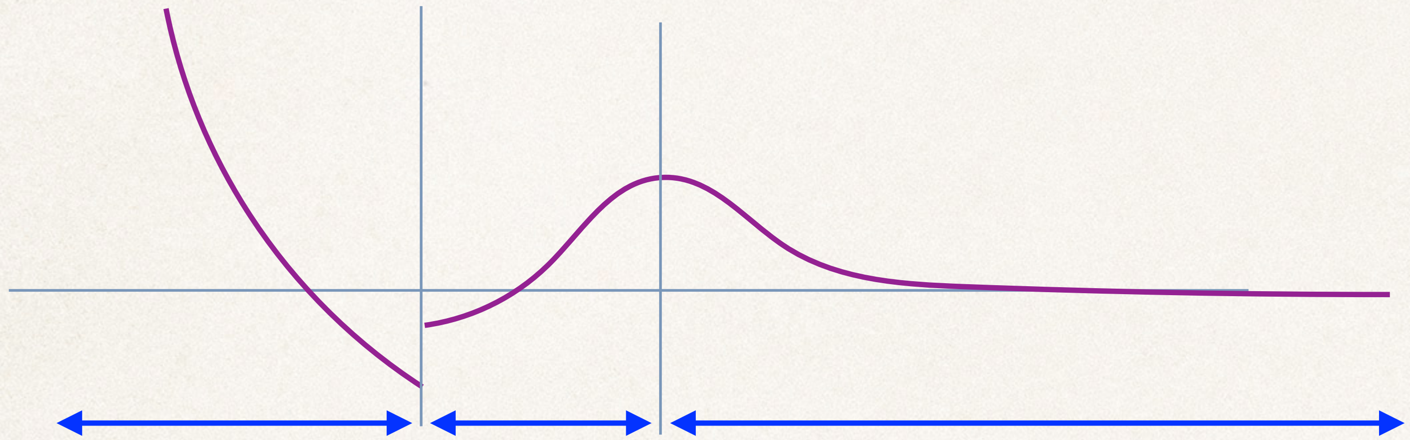

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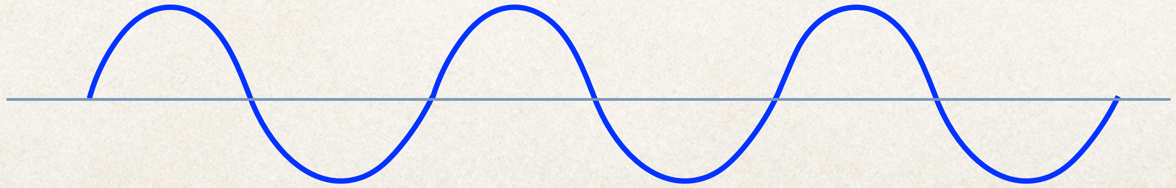
- Periodic functions $f(x+n) = f(x)$ are never tame (when not constant)

$\sin(x), x \in \mathbb{R}$



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- Period map is definable in $\mathbb{R}_{\text{an,exp}}$.

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Theorem [Cattani,Deligne,Kaplan '95]: For integer $\ell > 0$, locus of integral Hodge classes

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- original proof uses Hodge theory: nilpotent orbit theorem [Schmid]
  Sl(2) orbit theorem [Cattani,Kaplan,Schmid]

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→ relies on holomorphicity: often absent in physical situations

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  ⇒ works well for one-parameter limits [Schnell] [TG] '20, but too involved for multi-parameter limits

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$$\mathcal{S}_\ell = \Big\{ (x,v) \in E : \; v \in H_{\mathbb{Z},x} \text{ and } C_x v = v \text{ and } Q(v,v) = \ell \Big\}$$

is $\mathbb{R}_{\mathrm{an,exp}}$- definable, closed real-analytic subspace of E and the restriction of p to this set is proper with finite fibers.

- Proof: - tameness of Weil operator period map Φ

  - tameness of maps between bundles $\Phi_E : E \to \Gamma \backslash \big(G/K \times H_{\mathbb{C}} \big)$

  - lattice reduction: [e.g. Kneser]
    group Γ acts on set $\big\{ v \in H_{\mathbb{Z}} : Q(v,v) = \ell \big\}$ with finitely many orbits.

  - tameness of self-dual locus in a single orbit

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Question 1: What are the cycles associated to self-dual classes? (like in Hodge conjecture)

→ relevant in physics 'holography' [Lüst,Vafa,Wiesner,Xu]

Tadpoles and a new conjecture

- Condition $Q(v,v) = \ell$ (tadpole condition) is central to tameness
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- A new physics conjecture 'Tadpole conjecture': see also [Hebecker's talk] [Bena,Blaback,Graña,Lüst]…[Becker,Walcher,Wrase]

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Note: General evidence for loci realized in $\mathcal{M}$ where period map is sl(2)-orbit.

[Graña,TG,van de Heisteeg,Herraez,Plauschinn '22]

nilpotent orbit: [TG,Monnee] in progress

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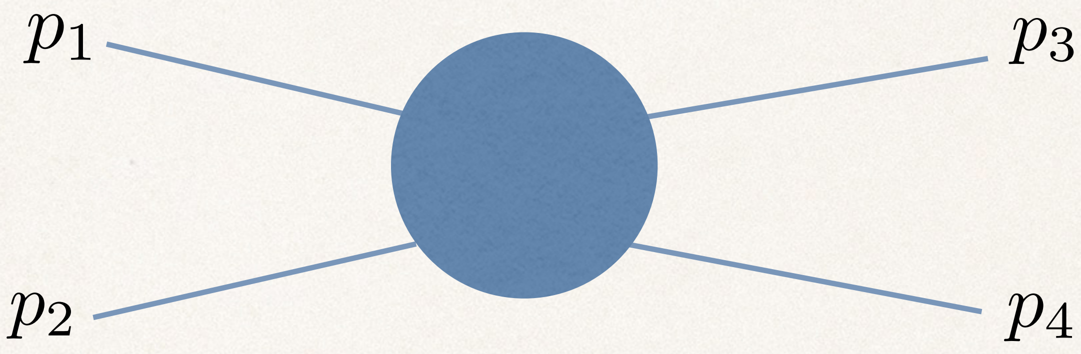
Question 2: Can one prove such a conjecture for $\mathcal{H}_\ell$ or $\mathcal{S}_\ell$?

Tameness in perturbative Quantum Field Theories

Perturbative QFTs

- Scattering amplitudes

external particles



$\mathcal{A}(p_i)$

How likely is the process?

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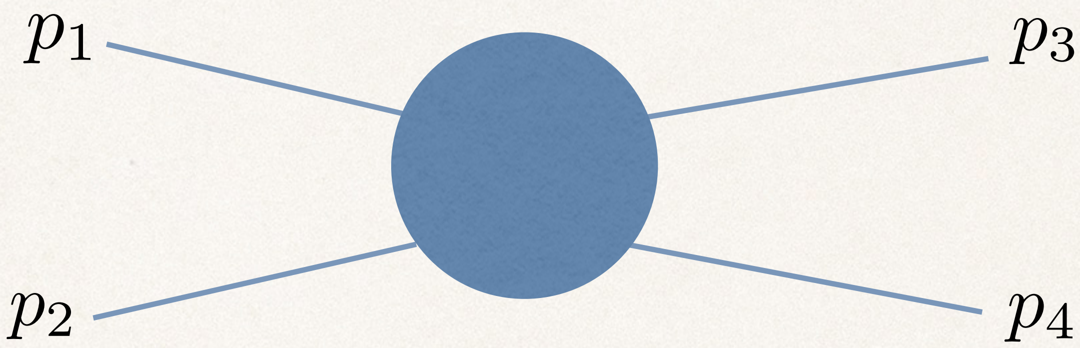

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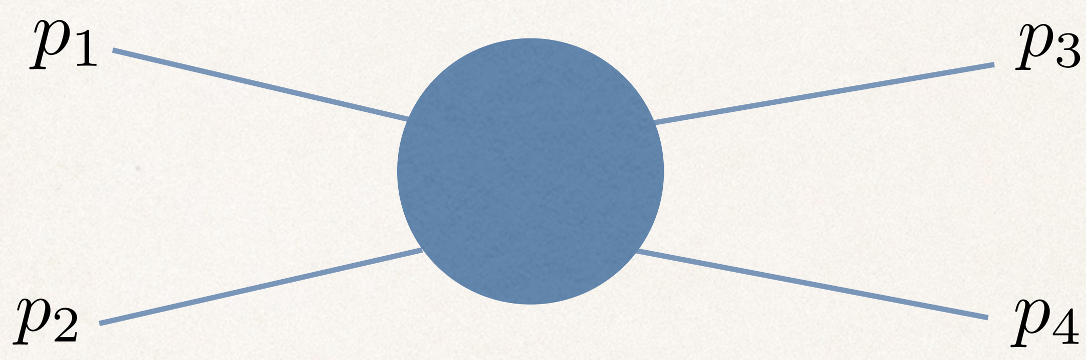

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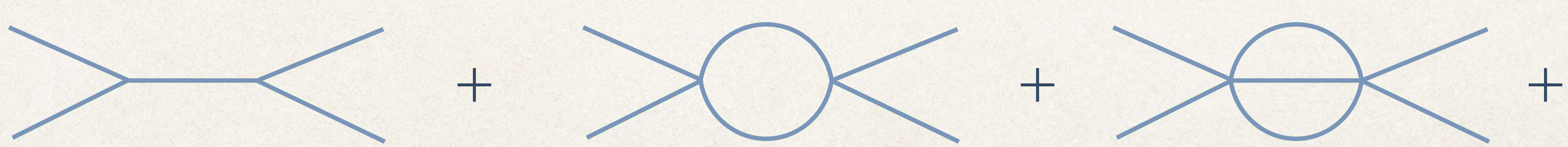
- Perturbative expansion: small coupling expansion $\lambda \ll 1$



$\rightarrow$ summing till fixed loop number: finite number of Feynman integrals

Perturbative QFTs

- Theorem*: For any renormalizable QFT with finitely many particles and interactions all finite-loop amplitudes are tame functions of the masses, external momenta, and coupling constants. [Douglas,TG,Schlechter '22]

*physics style

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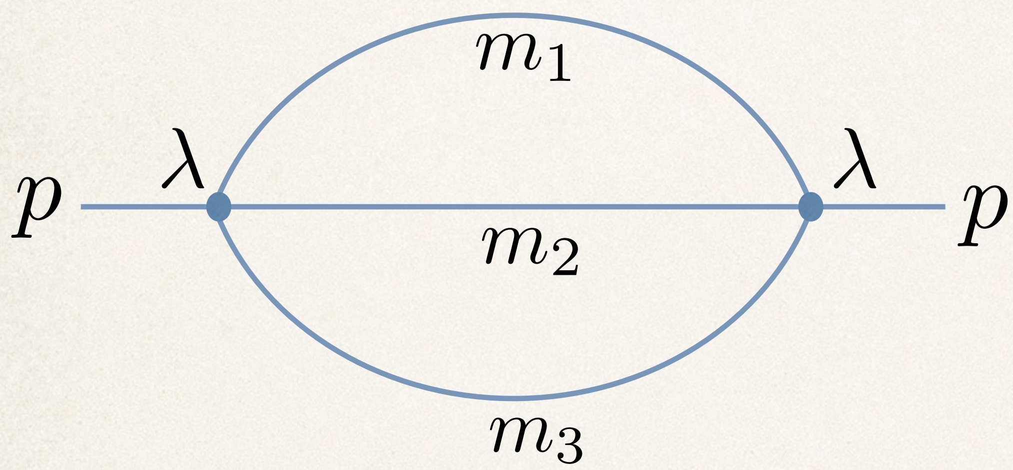

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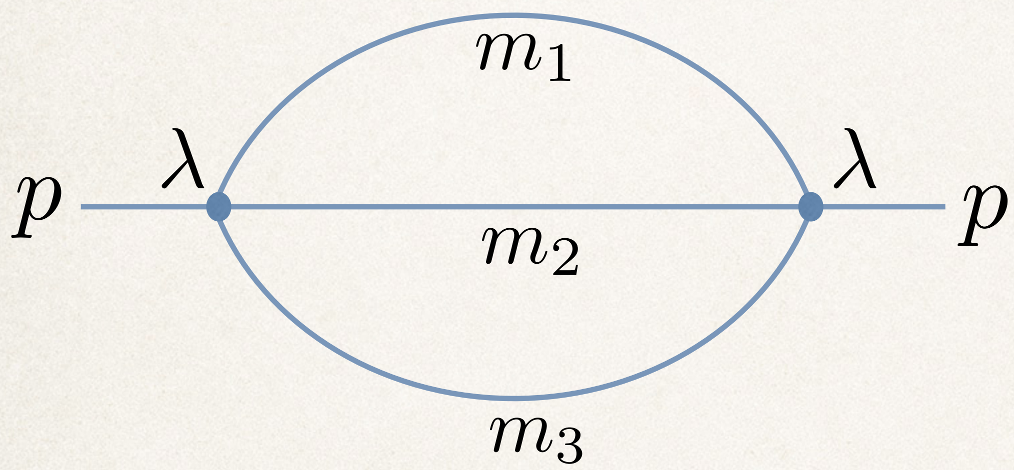

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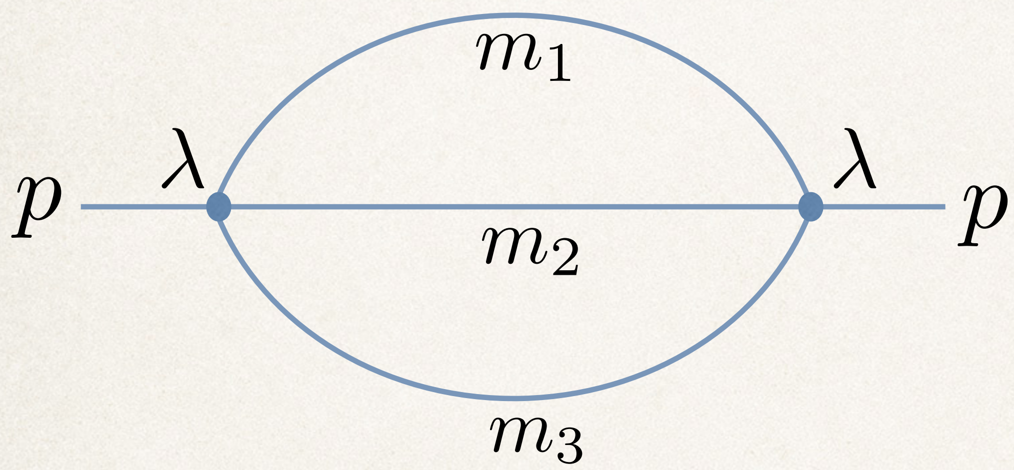

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Remarks: - theorem is non-trivial: interesting implications for Feynman amplitudes (symmetry $\longleftrightarrow$ relations) [in progress]

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Tameness of (relative) periods → Tameness of Feynman integrals

Tameness in non-perturbative Quantum Field Theories

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$$\langle \mathcal{O}_1(y_1)...\mathcal{O}_k(y_k)\rangle_\lambda$$

$\rightarrow$ complicated function on product of space-time $\Sigma \times ... \times \Sigma$ and parameter space $\mathcal{P}$

Simplest example

- Consider in 0d: $S = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \quad \rightarrow \quad Z = \sqrt{\frac{3}{\lambda}} e^{\frac{3m^4}{4\lambda}} \, m \, K_{\frac{1}{4}}\left(\frac{3m^4}{4\lambda}\right)$

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$\rightarrow$ well-defined notion of complexity for physical systems

[TG,Schlechter, van Vliet] to appear

Challenges in mathematics

- QFTs on finite lattice

correlation functions in 0d are ordinary integrals

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Note: Theorem for $\mathcal{S} = \mathbb{R}_{\text{an}} \rightarrow \widetilde{\mathcal{S}} = \mathbb{R}_{\text{an,exp}}$. [Comte,Lion,Rolin]

However, for non-perturbative results, we need exponential to be in $\mathcal{S}$.

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$\Rightarrow$ math. conjecture implies: [Douglas,TG,Schlechter '23]

in order that physical observables $\langle \mathcal{O}_1 ... \mathcal{O}_n \rangle_\lambda$ are tame functions of parameters λ one needs to require:

$S^{(0)}(\phi, \lambda)$ is tame function of λ, ϕ

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special, but interesting case: exponential periods with parameters

$$\Pi(\lambda) = \int_\Gamma e^{-f(\lambda)} \omega(\lambda)$$

$f(\lambda)$ algebraic function
$\omega(\lambda)$ algebraic differential form

Question 3: Are exponential periods definable in $\mathcal{P}(\mathbb{R}_{\mathrm{an,exp}})$?

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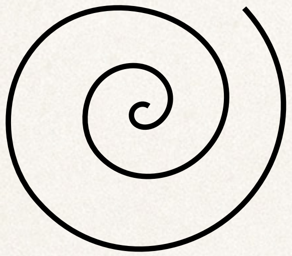

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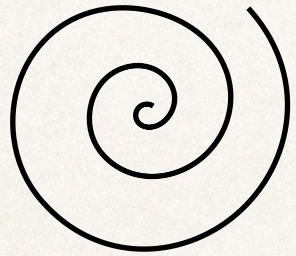

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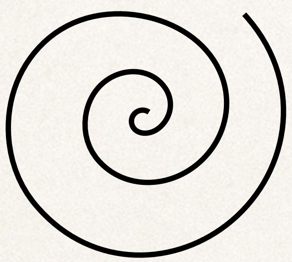

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- in general: tameness depends on the UV origin of the theory

Mapping out the tame parts of physics

- Tameness of Conformal Field Theory: precise conjectures

[Douglas,TG,Schlechter '23]

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(2) Space of CFTs is tame set under certain conditions (e.g. bound on degrees of freedom).

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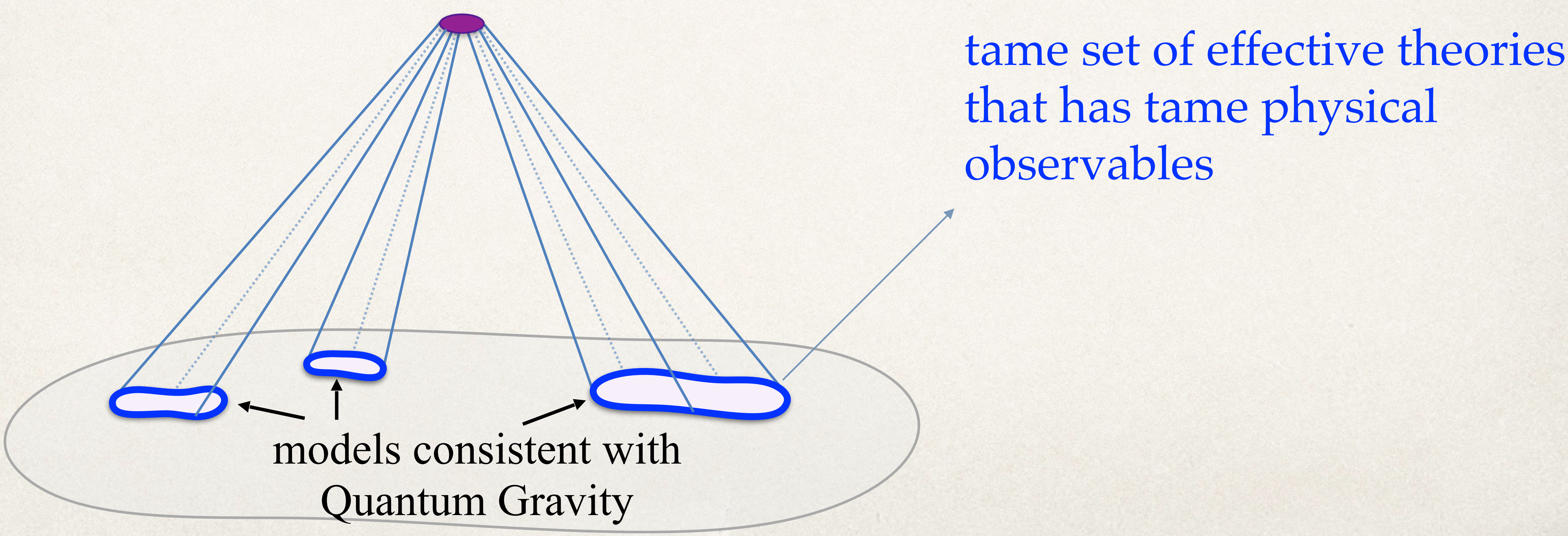
- Tameness of effective theories compatible with Quantum Gravity



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Legendre function of second kind not tame in the mass $m \rightarrow \infty$

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  [Banks,Dixon][Banks,Seiberg]

- Klein-Gordon field of mass m:

  → in AdS space: propagator $O_{\mathrm{AdS}_{d+1}}(y_1, y_2) \propto Q^{(d-1)/2}_{\sqrt{d^2/4+m^2}-1/2}(y_1 y_2)$

  Legendre function of second kind not tame in the mass $m \to \infty$

  Solutions: introduce cut-off $m \leq \Lambda_{\mathrm{UV}}$