The GAGA Theorems: The beutiful relationship between Algebraic Geometry and Analytic Geometry

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1 Introduction

With this text we aim to make a short presentation of the three theorems proven by Serre in his famous paper "Géométrie algébrique et géométrie analytique", also know as GAGA [Serre, 1956], which is our main reference.

What does Serre prove in this paper? He awnser very natural questions: if X is an algebraic variety over $\mathbb{C}$, there is a natural way to think of it as an analytic variety. How are these two structures related? When is an analytic variety algebraic? And some other similar questions.

More technically, the idea is to relate coherent sheaves on algebraic varieties to coherent sheaves on analytic varieties. As, roughly speaking, coherent sheaves encode the "geometry" of the varieties, its natural to think that if coherent shaves are related, the geometries must also be related.

In the end, Serre proved that coherent shaves over an algebraic varieties correspond to coherent sheaves over the same variety considered with an analytic structure, that they have the same cohomology and that they also have the same morphisms. This is the content of Theorems 4.1,4.2,4.3.

Among the applications, we have simple proof of a famous and surprising theorem due to Chow: "every analytic subvariety of the projective space is algebraic".

This text is organized in 4 sections. Section 2 contains some preliminaries and basic definitions. We hope that the reader is familiar with most of the concepts written there and that this section can work as a way of fixing notation. In the end of Section 2, we present some algebraic results which will be useful in some parts of the text.

Section 3 contains the main constructions necessary to understand GAGA, for example the analytic structure of an algebraic variety, the analytic sheaf associated to an algebraic sheaf, etc. There are also some results which are important in the proofs later, as the fact that the local ring of holomorphic functions and the one of regular functions have the same completion.

Section 4 contains the statements and proofs of the three main theorems of the paper and necessary lemmas.

Section 5, the last one, contains two interesting applications for GAGA: the Chow's theorem cited above and the proof that Betti numbers have an algebraic character.

We hope that the it is possible to understand the main ideas behind GAGA and see that it is not as complicated as some may think!

2 Preliminaries: Sheaves, Analytic Varities and Algebraic Varieties

In this section, we have the goal to present some basic properties and definitions we are considering and the text. We go through basic definitions of analytic and algebraic varieties and also basic theory o coherent sheaves. The main references for this part are [Serre, 1955], [Hartshorne, 1977], [Gunning and Rossi, 1965] and [Movasati, 2021]. Also, some results are taken directly from Serre's GAGA [Serre, 1956].

2.1 Sheaves

Definition 2.1. *Let X be a topological space. A* ***presheaf*** *(of abelian groups) $\mathscr{F}$ over X associates to each open subset $U \subset X$ an abelian group $\mathscr{F}(U)$ such that:*

- *if $V \subset U$, there is a restriction map $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$*
- *the restriction maps commute, that is, if $W \subset V \subset U$, $\rho_{UV} \circ \rho_{VW} = \rho_{UW}$*

We denote $\rho_{UV}(s) = s|_V^U$. In general, we drop U from the notation, being clear from the context which set we are restricting from.

Definition 2.2. *A* ***sheaf*** *$\mathscr{F}$ over X is a* ***presheaf*** *over X satisfying:*

- *if $U = \bigcup_{i \in I} U_i$ and $s \in \mathscr{F}(U)$ satisfy $s|_{U_i} = 0$, for all $i \in I$, then $s = 0$.*
- *if $U = \bigcup_{i \in I} U_i$ and we have sections $s_i \in \mathscr{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists a section $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$.*

The idea of sheaves is to generalize the concept of spaces of functions. That is the reason the maps are called restrictions and we ask for those properties. Actually, the idea is to consider spaces of functions which are determined by local properties, such as holomorphic functions, C^∞ functions, regular functions etc.

Of course, we do not need to be restricted to abelian groups. The same definition can be made considering rings, vector spaces, modules, etc.

Definition 2.3. *A morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of (pre)sheaves is simply a family of morphims $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ for any open subset U.*

We say two sheaves are isomorphic if there exists a morphism between them which is invertible, that is, each coordinate turns out to be isomorphism. It seems natural to define monomorphisms and epimorphisms in the same way. Although it will work for monomorphisms, extra caution is necessary for epimorphisms.

Proposition 2.4. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a sheaf morphism. Then the presheaf $ker\varphi$ defined by $ker\varphi(U) = ker\varphi_U$ is a sheaf.*

The same does not hold for the image, that is, the presheaf given by $\mathrm{Im}\varphi$ defined by $\mathrm{Im}\varphi(U) = \mathrm{Im}\varphi_U$ is not a sheaf.

In order to deal with this problem, we have a natural a way to contruct sheaves from presheaves. For this, we need a important concept when working with functions: germs. We can generalize this idea for any sheaf.

Definition 2.5. *Let $\mathcal{F}$ be a sheaf over X. For each point $x \in X$, we define the stalk $\mathcal{F}_x$ to be the set of equivalence classes of pairs (s, U) where $s \in \mathcal{F}(U)$ and $(s, U) \sim (t, V)$ if there is an open subset $W \subset U \cap V$ such that $s|_W = t|_W$.*

It is easy to prove that the stalks form abelian groups (resp rings, vector spaces etc). With this in hands we can state the following proposition.

Proposition 2.6. *Let $\mathcal{F}$ be a presheaf. Consider the presheaf $\hat{\mathcal{F}}$ to be*

$$\hat{\mathcal{F}}(U) = \prod_{x \in U} \mathcal{F}_x.$$

We have a natural morphism $\tau : \mathcal{F} \to \hat{\mathcal{F}}$ given by $s \mapsto (s_x)$. We define the sheaf associated to $\mathcal{F}$ to be:

$$\mathcal{F}^+(U) := \left\{ s \in \hat{\mathcal{F}}(U) \middle| \ \forall x \in U \ \exists V \ni x \text{ with } s|_V \in Im(\tau_V) \right\}$$

This sheaf satisfy a universal property. For any sheaf $\mathcal{G}$ and a morphism of presheaves φ, there is a unique morphism of sheaves between $\mathcal{F}^+$ and $\mathcal{G}$ which makes the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\tau} & \mathcal{F}^+ \\ & \underset{\varphi}{\searrow} & \downarrow \\ & & \mathcal{G} \end{array}$$

comutes.

Now, it is easy to define kernels and cokernels of morphisms of sheaves, taking the associated sheaf. This gives us a natural definition or kernel, image, epimorphism, monomorphism and exact sequences. We have:

Proposition 2.7. *If $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are sheaves, we have:*

a) $\varphi : \mathcal{F} \to \mathcal{G}$ is an isomorphism (all maps are invertible) if $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism for any $x \in X$.

b) $0 \to \mathcal{H} \to \mathcal{F} \to \mathcal{G} \to 0$ is exact if and only if $0 \to \mathcal{H}_x \to \mathcal{F}_x \to \mathcal{G}_x \to 0$ is exact.

We finish this subsection on sheaves defining ringed spaces, which will be the main objects necessary to define varieties.

Definition 2.8. *Let X be a topological space and $\mathcal{O}$ be a sheaf of rings over X. We call the pair $(X, \mathcal{O})$ a ringed space. If the stalks of $\mathcal{O}$ are local rings, we call $(X, \mathcal{O})$ locally ringed space.*

A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ togheter with a morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$, where the sheaf $f_*\mathcal{O}_X$ is given by $f_*\mathcal{O}_X(U) := \mathcal{O}_X(f^{-1}(U))$.

For locally ringed spaces, we ask the morphsm between the sheaves to preserve the maximal ideals.

2.2 Algebraic Varieties

Definition 2.9. *Let k be an algebraically closed field. We denote $k^n := \mathbb{A}^n_k$ and call it affine space. If k is fixed, we drop k from the notation. We define a topology on $\mathbb{A}^n$ by declaring the sets given by zeros of polynomials $f \in k[x_1, \ldots, x_n]$ to be closed. To get a structure of ringed space on $\mathbb{A}^n$, we consider the sheaf $\mathcal{O}$ whose stalks $\mathcal{O}_x$ are given by fractions of polynomials $\frac{P}{Q}$ with $Q(x) \neq 0$. In a open set, we consider the fractions with non zero denominator in that open set.*

With the affine space in hand, we can define an "affine algebraic variety": a locally closed subset $X \subset \mathbb{A}^n$ with $\mathcal{O}_X$ given by restriction of functions from $\mathcal{O}$.

The affine varieties are going to be the models for constructing general algebraic varieties. Notice that it is possible to define morphisms of algebraic varieties by restricting polynomial maps $\mathbb{A}^n \to \mathbb{A}^m$ and products of algebraic varieties considering the product embedded on a bigger affine space.

Definition 2.10. *A locally ringed space $(X, \mathcal{O}_X)$ is called an algebraic variety if it satisfy:*

1. *There is a finite covering $\mathscr{U} = \{V_i\}_{i\in I}$ of X such that each V_i (with the induced structure) is isormophic (as locally ringed spaces) to a locally closed subset of an affine space, ie, an affine variety.*

2. *The diagonal $\Delta \subset X \times X$ is closed.*

The most important example is, probably, the projective space:

Definition 2.11. *Let $\mathbb{P}^n$ be the quotient $\mathbb{C}^{n+1}/\mathbb{C}^\times$ given by projective coordinates $[x_0 : \cdots : x_n]$. Then $\mathbb{P}^n$ is an algebraic variety via the charts $V_i := \{x_i \neq 0\}$ which are isomorphic to the affine space. Any subvariety of the projective space is called projective variety.*

Many results about these objects can be proven and the reader can find general theory in the references cited in the beginning of this section. Through the text, we use the more common results and hope the reader will be able to find them in the refeences.

The same constructions as above can be made for the analytic case, if we consider the field to be $\mathbb{C}$ and holomorphic functions instead of polynomials.

Definition 2.12. *An analytic subvariety $X \subset \mathbb{C}^n$ is a subset which is locally given by holomorphic equations, that is, for each point $x \in X$ we can find a open set of $\mathbb{C}^n$ (with the usual topology) such that X is given by zeros of holomorphic functions on that open set. Closed analytic subvarities have a natural sheaf of holomorphic functions given by restrictions of holomorphic functions on $\mathbb{C}^n$. The sheaf of holomorphic functions is denoted by $\mathcal{H}_X$.*

It is important to notice that the local ring of germs $\mathcal{H}_X$ is notherian (as the ring of regular functions) and also satisfy a "nullstelensatz".

We can define a variety in almost the same way:

Definition 2.13. *A locally ringed space $(X, \mathcal{H}_X)$ is called analytic variety if:*

1. *There exists an open covering $\mathscr{U} = \{V_i\}_{i\in I}$ of X such that the V_i (taken with the restriction of the sheaf) are all isomorphic (as locally ringed spaces) to an analytic subset of $\mathbb{C}^n$.*

2. *X is Hausdorff.*

In the next section, we will see how to associate a structure of analytic variety to algebraic varieties over $\mathbb{C}$.

2.3 Coherent Sheaves

A very importante concept on algebraic geometry and complex geometry is the concept of coherent sheaf. It is actually a way to generalize vector bundles (which can be seen as sheaves whose stalks are the fibers of the vector bundle).

Definition 2.14. *Let $(X, \mathcal{O}_X)$ be a ringed space. A $\mathcal{O}_X$-module $\mathcal{F}$ is a sheaf over X such that, for each U, $F(U)$ has the structure of $O_X(U)$-module.*

In the case of an algebraic variety, we call the $\mathcal{O}_X$-modules *algebraic sheaves* and in the case of an analytic variety, we call the $\mathcal{H}_X$-modules *analytic sheaves.*

Definition 2.15. *A $\mathcal{O}_X-$module $\mathcal{F}$ is said to be of finite type if it is locally finitely genereted by sections, that is, for each $x \in X$, there exists an open neighborhood $x \in U \subset X$ and sections $s_1, \ldots, s_p \in \mathcal{F}(U)$ such that the corresponding germs $s_{1,x}, \ldots, s_{p,x} \in \mathcal{F}_x$ generate the stalk over $\mathcal{O}_{X,x}$*

It is easy to prove that if the sections s_i generate the stalk in a point, they generate all stalks of the neighborhood U.

Definition 2.16. *A sheaf is called coherent if it is of finite type and, besides that, any epimorphism $\mathcal{O}_X^q \to \mathcal{F} \to 0$ has kernel of finite type. In other words, we have, for $x \in X$ and $U \ni x$ and any morphism as above, an exact sequence:*

$$\mathcal{O}_X^p|_U \to \mathcal{O}_X^q|_U \to \mathcal{F}|_U \to 0.$$

Some important properties on coherent sheaves include:

Proposition 2.17. *We have:*

1. *If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence with two coherent sheaves, then the third is also coherent.*

2. *The kernel, cokernel and the image of a morphism between coherent sheaves is coherent.*

3. *The tensor product between two coherent sheaves is coherent.*

4. *$Hom(\mathcal{F}, \mathcal{G})_x \cong Hom(\mathcal{F}_x, \mathcal{G}_x)$, where the first space is the stalk of the sheaf associated to the presheaf $U \mapsto Hom(\mathcal{F}(U), \mathcal{G}(U))$*

For details, see [Serre, 1955]. Some other results may be used through the text.

2.4 Flat Pairs and Completions

In this subsection, we have the goal to present some algebraic results which will be important to understand GAGA. Most of the facts presented in this subsection come from the appendix of [Serre, 1956].

Definition 2.18. *Let R be a ring and let A be an R-module. We say that A is R-flat (or simply flat) if, for any exact sequence:*

$$0 \to B \to C \to D \to 0$$

of R-modules, we have that the sequence:

$$0 \to A \otimes_R B \to A \otimes_R C \to A \otimes_R D \to 0$$

is also exact.

Notice that, by definition of the Tor functor, this is the same as having $Tor(A, \bullet) = 0$.

If $A \subset B$ is a inclusion of rings, we say that (A, B) is flat pair if the A-module B/A is flat.

Proposition 2.19. *Let $A \subset B$. (A, B) is a flat pair if and only if B is A-flat and one of the following is true:*

(i) For any A-module (of finite type) E, the morphism $E \to E \otimes_A B$ is injective.

(ii) For any ideal $I \subset A$, we have that $I \cdot B \cap A = I$.

Proof. Consider the exact sequence:

$$0 \to A \to B \to B/A \to 0.$$

It induces a long exact sequence after tensoring by E over A:

$$Tor(A, E) \to Tor(B, E) \to Tor(B/A, E) \to A \otimes E \to B \otimes E$$

As $A \otimes E = E$ and $Tor(A, E) = 0$, we get:

$$0 \to Tor(B, E) \to Tor(B/A, E) \to E \to B \otimes E$$

Now, B/A is A-flat (that is, $Tor(B/A, E) = 0$) if and only if $E \to B \otimes E$ is injective and $Tor(B, E) = 0$, which show (i).

For condition (ii), we simply take $E = A/I$. □

Lemma 2.20. *If $A \subset B \subset C$ such that (A, C) is flat and (B, C) is flat, then (A, B) is flat.*

Proof. We will use proposition 2.19. First, we show that B is A-flat. Let $0 \to E \to F$ be an injection of A modules. Let N be the kernel of the morphism $E \otimes_A B \to F \otimes_A B$. As C is B-flat, we have that the sequence

$$0 \to N \otimes_B C \to (E \otimes_A B) \otimes_B C \to (F \otimes_A B) \otimes_B C$$

is exact.

Now, using that the tensor product is associative, we get the sequence:

$$0 \to N \otimes_B C \to E \otimes_A C \to F \otimes_A C.$$

As C is A-flat, we conclude that $N \otimes BC = 0$. By 2.19 (i) and the fact that (B, C) is flat, we conclude that $N = 0$. This implies B is A-flat.

Condition (ii) is trivial: if $I \subset A$, we have

$$I \subset I \cdot B \subset I \cdot C$$

and therefore

$$I = I \cap A \subset I \cdot B \cap A \subset I \cdot C \cap A = I,$$

since (A, C) is a flat pair.

□

We can now study flatness in the case of local rings and their completions: the case for which we are going to apply the results presented here. Assume A is local notherian ring and let $\mathfrak{m}$ be its maximal ideal.

Let E be a finite generated module over A. Recall that the $\mathfrak{m}$-adic topology on E is the topology for which the submodules $\mathfrak{m}^n E$ form a local basis around $0 \in E$ and submodules $r + \mathfrak{m}^n E$ form a local basis around any other $r \in E$.

This topology is metrizable and the its completion is denoted by $\hat{E}$. It has a natural structure of $\hat{A}$-module with the operations extended by continuity.

We can define a natural morphism $E \otimes_A \hat{A} \to \hat{E}$ extending the injection $E \to \hat{E}$ by linearity.

Proposition 2.21. *With the same notation as above, the morphism $E \otimes_A \hat{A} \to \hat{E}$ is bijective.*

Proof. As E is finitely generated, we can write an exact sequence:

$$0 \to R \to L \to E \to 0$$

where L is a free module of finite rank. As L is notherian, R is also finitely generated.

Tensoring and taking the completions, we get a commutative diagram:

$$\begin{array}{ccccccc} R \otimes_A \hat{A} & \to & L \otimes_A \hat{A} & \to & E \otimes_A \hat{A} & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \hat{R} & \to & \hat{L} & \to & \hat{E} & \to & 0 \end{array}$$

As the middle arrow is clearly an isomorphism ($L \otimes_A \hat{A} = \hat{A}^r = \widehat{A^r} = \hat{L}$), we conclude that the right arrow must be surjective.

So the morphism $F \otimes_A \hat{A} \to \hat{F}$ is surjective for any F. Therefore we have that the left arrow in the diagram is also surjective. Applying the five lemma, we conclude the right arrow must be an isomorphism. □

We are now ready to prove the main theorems of this subsection:

Theorem 2.22. *Let A be local ring and let $\hat{A}$ be its completion. Then the pair $(A, \hat{A})$ is flat.*

Proof. Following proposition 2.19 we first notice that $\hat{A}$ is A-flat. Indeed, if $0 \to E \to F$ is an injection, we have that $0 \to \hat{E} \to \hat{F}$ is also an injection. Using the isomorphism of 2.21, we conclude that $0\hat{E} \otimes_A \hat{A} \to F \otimes_A \hat{A} \to 0$ remains exact and thus $\hat{A}$ is A-flat.

In order to finish the proof, we verify assertion (i) from 2.19: $E \to \hat{E}$ is injective if E is finitely generated, but $\hat{E} = E \otimes_A \hat{A}$. □

Theorem 2.23. *Let $\theta : A \to B$ be a morphism. Then θ extends by continuity to a morphism $\hat{\theta} : \hat{A} \to \hat{B}$. If $\hat{\theta}$ is bijective, then θ is injective and the couple (A, B) (where A is identified as a subring of B via θ) is flat.*

Proof. The assertion that θ is injective is obvious since it is a restriction of $\hat{\theta}$. For the second part, consider the inclusions $A \subset B \subset C := \hat{B} = \hat{A}$. By 2.22, the pairs (A, C) and the pair (B, C) are flat. It remains to apply 2.20. □

The result above apply for quotients of A, that is, if I is an ideal of A and $J = \theta(I)B$ in B, the quotients form a flat pair.

3 Towards the Main Theorems: First relations between Algebraic and Analytic Varieties

As we introduced last section, algebraic varieties are naturally endowed with the so called Zariski topology and analytic varieties are equipped with the natural "usual" or "strong" topology from its charts.

Suppose X is an algebraic variety over $\mathbb{C}$. As affine algebraic varieties are obviously analytic (since polynomials are holomorphic), we have two topologies over X: the Zariski topology, from its structure of algebraic variety, and a topology induced from the usual topology of its affine charts.

As Serre does in his article , we add the letter Z at the beggining of the words when refering to Zariski topology. For example, Z-open means open in Zariski topology.

3.1 The Analytic Variety associated to an Algebraic Variety

Our goal here is to put a structure of analytic variety in any algebraic variety. The first step is to establish some results about the Zariski and usual topology of $\mathbb{C}^n$.

Lemma 3.1. *The following is true:*

a) The usual topology of $\mathbb{C}^n$ is stronger than its Zariski topology.

b) All Z-locally closed subsets of $\mathbb{C}^n$ are analytic.

c) If U and U' are two Z-locally closed subsets of $\mathbb{C}^n$ and $\mathbb{C}^m$ and $f : U \to U'$ is a regular map, then f is holomorphic.

d) If, in the above situation, f is an algebraic (birregular) isomorphism, then f is a biholomorphism.

Proof. We prove each of the assertions separately:

a) As all polynomials are continous in the usual topology, zeros of polynomials, i.e Z-closed sets, are closed. So, every element of the Z-topology is an element of the usual topology.

b) If F is Z-locally closed, there is an Z-open subset U of $\mathbb{C}^n$ and a regular function φ in U such that F is the zero set of φ. As φ is regular, for each point of F, φ can be written as quotient of polynomials with non zero denominator. Thus, φ is holomorphic in this neighborhood and F is locally given by its zeros. We conclude that F is analytic.

c) A regular map $f : U \to U'$ is simply a polynomial map $F : V \subset \mathbb{C}^n \to W \subset \mathbb{C}^m$ restricted to U and U', where U and W are Z-open subsets. As all polynomial maps are holomorphic, and the holomorphic maps on F are restrictions of holomorphic maps in open sets containg U and U', we have the result.

d) Just apply 3 for the inverse.

□

With this lemma in hands, we can equip any algebraic variety over $\mathbb{C}$ with a structure of analytic variety:

Proposition 3.2. *Let X be an algebraic variety over $\mathbb{C}$. There is a unique structure of analytic variety over X such that, for each affine chart $\varphi : V \to U$ from a Z-open subset of X to a Z-locally closed subset of $\mathbb{C}^n$, φ is an analytic isomorphism from V (with the induced topology) to U (with its natural analytic structure from 3.1b).*

Proof. We need to define a topology and a sheaf $\mathcal{H}_X$ on X. Let $X = \bigcup_{i \in I} V_i$ an affine covering. Let φ_i be the affine charts.

Notice that if we have two charts φ and ψ, the transition maps $\varphi \circ \psi^{-1}$ are algebraic isomorphisms and by Lemma 3.1d, they are analytic isomorphisms. This allow us to make the following definition:

$$W \text{ is open if } W \cap V_i \text{ is open for all } i$$

$$\mathcal{H}_{X,x} = \mathcal{H}_{\varphi_i(V_i),\varphi_i(x)} \text{ for } x \in V_i \text{ and } \mathcal{H}_X|_{V_i} = \varphi^* \mathcal{H}_{\varphi(V_i)}$$

where $\mathcal{H}_{\varphi_i(V_i)}$ is the sheaf of holomorphic functions on $\varphi(V_i)$

Notice that $\mathcal{H}_X$ is well defined, since it is defined locally. Obviously this induces analytic chats on X.

To see that X is Hausdorff, take two points $x, y \in X$. If they are on the same chart, we can separate them, since U_i is a subset of $\mathbb{C}^n$. Otherwise, recall that, by definition of algebraic variety, the image T_{ij} of the map $V_i \cap V_j \to U_i \times U_j$, $x \mapsto (\varphi_i(x), \varphi_j(x))$ is Z-closed in $U_i \times U_j$ and 3.1a implies that it is closed. So, if $x \in V_i$ and $y \in V_j$ and both are not in the intersection, we get that $(\varphi_i(x), \varphi_j(y)) \notin T_{ij}$. By closedness, it is possible to find an open set of $U_i \times U_j$ that does not intersect T_{ij}. As the usual topology of the product is the actual product topology, we can find open sets A_i with $V_i \supset A_i \ni x$ and A_j with $V_j \supset A_j \ni y$ such that $\varphi(A_i) \times \varphi(B_i)$ does not intersect T_{ij}. But not intersecting T_{ij} means exactly that $\varnothing = V_i \cap V_j \cap A_i \cap A_j = A_i \cap A_j$.

This shows that X has a structure of analytic variety. The uniqueness follows from the fact that the identity would induce an biholomorphism, since it is locally a biholomorphism (as the structures are the same on the charts) and a global bijection. □

From now on, we denote by X^h the analytic variety associated to X by the proposition above. We can also conclued some simple properties of the space X^h from the definition.

Lemma 3.3. *If X is an algebraic variety, $Y \subset X$ is a Z-locally closed subset and $f : X \to Z$ is a regular map, we have:*

a) $(X \times Z)^h = X^h \times Z^h$

b) Y^h is an analytic subvariety of X^h and the anlytic structure of Y^h coincides with the induced structure of $Y \subset X^h$

c) f induces a map from X^h to Y^h and this map is holomorphic.

3.2 Local Rings

Now we proceed to the task of relating the algebraic local ring of regular germs on X and the analytic local ring of holomorphic germs on X^h. Let us denote by $\mathcal{O}_X$ the sheaf of regular functions on X and by $\mathcal{H}_X$ the sheaf of holomorphic functions on X^h.

As all regular functions are holomorphic, we can define a map in the level of germs $\theta : \mathcal{O}_{X,x} \to \mathcal{H}_{X,x}$. Note that it is not possible to define a morphism of sheaves, since X^h and X have different topologies! Also, as θ is local, that is, it takes the maximal ideal of $\mathcal{O}_{X,x}$ to the maximal ideal of $\mathcal{H}_{X,x}$ (since both are given by functions which are zero at x), we can extend it to the completions $\hat{\theta} : \hat{\mathcal{O}}_{X,x} \to \hat{\mathcal{H}}_{X,x}$.

Theorem 3.4. *The map θ satisfy the following:*

a) The morphism $\hat{\theta}$ is an isomorphism.

b) If Y is a Z-locally closed subset of X, and $J = J_x(Y)$ is the ideal of germs of regular functions vanishing on Y, the image $\theta(J)$ generates the ideal $A = A_x(Y)$ of germs of holomorphic functions vanishing on Y^h.

Proof. We divide the proof in two steps. The general case and the case $X = \mathbb{C}^n$.

Step 1 Let $X = \mathbb{C}^n$ and $x = 0$. In this case, the first part of the theorem is trivial, since both the ring of convergent power series (analytic functions) and the ring of polynomials have the same completion (which is the ring of formal power series). The map θ is simply the inclusion in this case and, therefore, induces the identity in the completion.

For the second part, we fix the notation $I = \langle \theta(J) \rangle$ and let $f \in A$ be any holomorphic function. By the holomorphic Nullstelesatz, we know $f^r \in I$ for some $r > 0$, since $V(I) = V(A) \implies \sqrt{I} = A$. Passing to the completion:

$$f^r \in \hat{I} = I \cdot \hat{\mathcal{H}}_{X,x} = J \cdot \hat{\mathcal{O}}_{X,x} = \hat{J}$$

Now, using that J is radical and that $\hat{J}$ is also radical (since we are working over excellent rings), we conclude that $f^r \in \hat{J} \implies f \in \hat{J}$. Thus, $f \in \mathcal{H}_{X,x} \cap \hat{J} = \mathcal{H}_{X,x} \cap \hat{I} = I$. We conclude $I = A$.

Step 2 For the general case, as the situation is local, we can assume that $\mathcal{H}_{X,x} = \frac{\mathcal{H}_{\mathbb{C}^n,0}}{A}$ and that $\mathcal{O}_{X,x} = \frac{\mathcal{O}_{\mathbb{C}^n,x}}{J}$ for some ideal J and A. As, locally, functions on X are restrictions of functions on the affine space, we conclude that the map $\theta = \theta_X$ for X is the map $\eta = \theta_{\mathbb{C}^n}$ after passing to the quotient. Using step 1, we get that $\hat{\eta}$ is an isomorphism and $\eta(J)$ generates A and, therefore, $\hat{\eta}(\hat{J}) = \hat{A}$. Now, since

$$\hat{\mathcal{H}}_{X,x} = \frac{\hat{\mathcal{H}}_{\mathbb{C}^n,0}}{\hat{A}}$$

and

$$\hat{\mathcal{O}}_{X,x} = \frac{\hat{\mathcal{O}}_{\mathbb{C}^n,x}}{\hat{J}}$$

we have that θ induces an isomorphism (both ideals and rings are the same above!).

The second part follows from the fact that the ideal of a subvariety Y is simply the image of the ideal of Y as a subvariety of $\mathbb{C}^n$ when we pass to the quotient.

$\square$

Corollary 3.5. *θ is injective.*

Corollary 3.6. *The rings $\mathcal{O}_{X,x}$ and $\mathcal{H}_{X,x}$ have the same dimension. Therefore, the analytic dimension is the same as the algebraic dimension.*

Corollary 3.7. *The pair $(\mathcal{O}_{X,x}, \mathcal{H}_{X,x})$ is flat.*

3.3 Zariski Topology, Strong Topology and Morphisms

We start this section by stating some topological results relating the two topologies of X. After we will try to see when we can say that an analytic morphism is algebraic.

Proposition 3.8. *Let X be an algebraic variety and $U \subset X$ a Z-open and Z-dense subset of X. Then U is dense in X^h*

Proof. Let $Y = X - U$. If U was not dense, we would be able to find $y \in Y$ and a neighborhood $V \ni y$ such that $V \subset Y$. This would mean that, at the point y, the local rings $\mathcal{H}_{Y,y}$ and $\mathcal{H}_{X,x}$ are isomorphic. In particular, we would have $A_x(Y) = 0$. Using Corollary 3.5, this would imply that $J_x(Y) \subset \mathcal{O}_{X,x}$ is zero. But this would mean that U is not Z-dense: a contradiction. □

Proposition 3.9. *X is complete if and only if X^h is compact.*

This proof is a consequence of the Chow's lemma, which can be found on [Hartshorne, 1977].

After these topological facts, we state a strong theorem relating analytic and regular maps. We know, of course, that not all holomorphic maps are regular. Although, we can find nice conditions in which holomorphic maps are actually regular.

Theorem 3.10. *Let X and Y be two algebraic varieties and $f : X \to Y$ be a bijective regular map. If f is a biholomorphism from X^h to Y^h, then f is a birregular isomorphism.*

This theorem can be applied immediatly to obtain a criterion for regularity.

Corollary 3.11. *Let X and Y be two algebraic varieties and $f : X^h \to Y^h$ be an holomorphic map. If the graph T is Z-locally closed in $X \times Y$, then f is a regular morphism.*

Proof. The corollary follows from 3.10. Just consider the composition $X \to \Gamma(f) \subset X \times Y \to Y$. We have that the second map is regular, as it is the projection. The first one is an biholomorphism, since it is holomorphic and its inverse is the projection, which is regular. By the theorem 3.10, the first map is a birregular isomorphism and therefore the composition is regular. □

3.4 Analytification of Sheaves

After explaining how an algebraic variety can be made into an analytic variety, we need to explain how algebraic sheaves can be turned into analytic sheaves. In less technical words, we are now going to see how geometric constructions on algebraic varieties can be transposed to analytic varieties.

Troughout this section, we will omit the letter X from the notation $\mathcal{H}_X$ and $\mathcal{O}_X$, writing only $\mathcal{H}$ and $\mathcal{O}$.

Definition 3.12. *Let $i : X^h \to X$ be the continous mapping given by the identity map (see 3.1) and let $\mathcal{F}$ be an algebraic sheaf ($\mathcal{O}$-module). We define the* **analytification of** *$\mathcal{F}$ to be the sheaf*

$$\mathcal{F}^h = i^*\mathcal{F} \otimes_{i^*\mathcal{O}} \mathcal{H},$$

where $i^*\mathcal{F}$ represents the pullback of the sheaf $\mathcal{F}$:

$$i^*\mathcal{F}(U) = \operatorname*{colim}_{\substack{V \supset i(U) \\ V \text{open}}} \mathcal{F}(V).$$

Notice that every algebraic morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of algebraic sheaves induces an analytic morphism:

$$\varphi : \mathcal{F}^h \to \mathcal{G}^h$$

between the analytifications (simply using that i^* and the tensor product are funtors). This means that the analyticfication is a functor.

Proposition 3.13. *The functor $\mathcal{F} \mapsto \mathcal{F}^h$ satisfies:*

a) $\mathcal{F} \mapsto \mathcal{F}^h$ is exact.

b) For any $\mathcal{F}$, the morphism $\alpha : i^*\mathcal{F} \to \mathcal{F}$ induced by the inclusion $i^*\mathcal{O} \hookrightarrow \mathcal{H}$ is injective.

c) If $\mathcal{F}$ is an algebraic coherent sheaf, $\mathcal{F}^h$ is an analytic coherent sheaf.

Proof. a) Let

$$0 \to \mathcal{A}' \to \mathcal{A} \to \mathcal{A}'' \to 0$$

be an exact sequences of sheaves over X.

Since i^* preserves the stalks, it preseves exactness.

Now, by 3.7, H_x is a flat O_x module and, therefore:

$$0 \to (i^*\mathcal{A}')_x \underset{\mathcal{O}_x}{\otimes} \mathcal{H}_x \to (i^*\mathcal{A})_x \underset{\mathcal{O}_x}{\otimes} \mathcal{H}_x \to (i^*\mathcal{A}'')_x \underset{\mathcal{O}_x}{\otimes} \mathcal{H}_x \to 0$$

is exact, as desired.

b) Consider the sequence:

$$0 \to i^*(\mathcal{O}) \to \mathcal{H}$$

At level of stalks:

$$0 \to \mathcal{O}_x \to \mathcal{H}_x \to \frac{\mathcal{H}_x}{\mathcal{O}_x} \to 0$$

By 3.7 we have that $\frac{\mathcal{H}_x}{\mathcal{O}_x}$ is a flat $\mathcal{O}_x$-module. Therefore $Tor\left(F_x, \frac{\mathcal{H}_x}{\mathcal{O}_x}\right) = 0$. Thus, the sequence above is exact after tensoring with $\mathcal{F}_x$. This implies $i^*\mathcal{F} \to \mathcal{F}^h$ is injective.

c) For this part, assume $\mathcal{F}$ is coherent as an $\mathcal{O}$-module. Than, for each $x \in X$, there exists numbers p, q such that

$$\mathcal{O}^p \to \mathcal{O}^q \to \mathcal{F} \to 0$$

is exact in a Z-neighborhood U of x.

By part a and the fact that the analytification of $\mathcal{O}$ is $\mathcal{H}$, we get an exact sequence:

$$\mathcal{H}^p \to \mathcal{H}^q \to \mathcal{F}^h \to 0$$

As the Zariski topology is weaker then the usual topology, we conclude that $\mathcal{F}^h$ is coherent.

□

Besides that, the functor also comutes with the extension by zeros:

Proposition 3.14. *Let Y be a Z-closed subvariety of X and $\mathcal{F}$ be a sheaf over Y. The sheaves $(\mathcal{F}^h)^X$ and $(\mathcal{F}^X)^h$ are naturally isomorphic, where $\mathcal{F}^X$ is the extension of $\mathcal{F}$ by zeros to X.*

Proof. We just need to check the fact in the stalks. We have, for $x \notin Y$, that both sheafs have zero stalks. It suffices to show that the stalks are isomorphic for $x \in Y$. We have:

$$(\mathcal{F}^X)^h_x = \mathcal{F}_x \underset{\mathcal{O}_{X,x}}{\otimes} \mathcal{H}_{X,x}$$

$$(\mathcal{F}^h)^X_x = \mathcal{F}_x \underset{\mathcal{O}_{Y,x}}{\otimes} \mathcal{H}_{Y,x}$$

If $J = J_x(Y)$, we have that $\mathcal{O}_{Y,x} = \frac{\mathcal{O}_{X,x}}{J}$. Now, by 3.4b, we have

$$\mathcal{H}_{Y,x} = \frac{\mathcal{H}_{X,x}}{J \cdot \mathcal{H}_{X,x}} \cong \mathcal{H}_{X,x} \underset{\mathcal{O}_{X,x}}{\otimes} \frac{\mathcal{O}_{X,x}}{J} \cong \mathcal{H}_{X,x} \underset{\mathcal{O}_{X,x}}{\otimes} \mathcal{O}_{Y,x}.$$

Therefore:

$$(\mathcal{F}^h)^X_x \cong \mathcal{F}_x \underset{\mathcal{O}_{Y,x}}{\otimes} \mathcal{H}_{Y,x} \cong \mathcal{F}_x \underset{\mathcal{O}_{Y,x}}{\otimes} \mathcal{H}_{X,x} \underset{\mathcal{O}_{X,x}}{\otimes} \mathcal{O}_{Y,x} \cong \mathcal{F}_x \underset{\mathcal{O}_{X,x}}{\otimes} \mathcal{H}_{X,x} \cong (\mathcal{F}^X)^h_x.$$

□

As every Z-open subset is open, if s is a section in $\Gamma(U, \mathcal{F})$, s induces a section on the pullback (in this case, as $i(U) = U$ is Z-open, $i^*\mathcal{F}(U) = \mathcal{F}(U)$. Therefore, $s \otimes 1$ is a section in $\Gamma(U, \mathcal{F}^h)$. This induces a homomorphism:

$$\begin{array}{rcl} \varepsilon : \Gamma(U, \mathcal{F}) & \to & \Gamma(U^h, \mathcal{F}^h) \\ s & \mapsto & s \otimes 1 \end{array}$$

where we use the notation U^h to stress the fact that we are considering the analytic structure.

If we take a finite Z-open covering of X, $\mathcal{U} = \{U_i\}$, this covering is also a covering of X^h, which we denote by $\mathcal{U}^h$ to stress that they cover different spaces. If we consider the Cech Complex with respect to this covering we get obvious morphisms defined the same way as ε in each coordinate:

$$\varepsilon : C(\mathcal{U}, \mathcal{F}) \to C(\mathcal{U}^h, \mathcal{F}^h).$$

As ε commutes with the coboundary operator δ, we obtain a map in the cohomology (after passing to the colimit):

$$\varepsilon : H^q(X, \mathcal{F}) \to H^q(X^h, \mathcal{F}^h).$$

This morphism ε is functorial, that is, commutes with morphisms $\mathcal{F} \to \mathcal{G}$. It also respects long exact sequences of coherent sheaves:

Proposition 3.15. *Let*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0,$$

be an exact sequence.

If $\mathcal{A}$ is coherent, the diagram:

$$\begin{array}{ccc} H^q(X, \mathcal{C}) & \xrightarrow{\Delta} & H^{q+1}(X, \mathcal{A}) \\ \varepsilon \downarrow & & \varepsilon \downarrow \\ H^q(X^h, \mathcal{C}^h) & \xrightarrow{\Delta} & H^{q+1}(X^h, \mathcal{A}^h) \end{array}$$

is commutative for any q.

Proof. □

The morphism ε is going to play a very important role. In particular, we are going to show that it is an isomorphism if X is projective. We left the statements of the three main theorems and their proofs to the next section.

4 The Three Main Theorems: Proofs

After the constructions and results from last section, we are ready to state and prove the main theorems. In the following, we assume X is a projective variety, that is, a Z-closed subset of the projective space $\mathbb{P}^n$

Theorem 4.1. *For any coherent algebraic sheaf $\mathcal{F}$ over X and any q, the morphism*

$$\varepsilon : H^q(X, \mathcal{F}) \to H^q(X^h, \mathcal{F}^h)$$

is an isomorphism.

Theorem 4.2. *If $\mathcal{F}$ and $\mathcal{G}$ are two algebraic coherent sheaves, every analytic morphism $\mathcal{F}^h \to \mathcal{G}^h$ is induced by an unique algebraic morphism $\mathcal{F} \to \mathcal{G}$.*

Theorem 4.3. *For any coherent analytique sheaf $\mathcal{M}$ over X^h, there exists a unique (up to isomorphism) algebraic coherent sheaf for which $\mathcal{F}^h$ is isomorphic to $\mathcal{M}$*

These theorems show that the theory of algebraic coherent sheaves coincides with the theory of analytic coherent sheaves. That is, the algebraic geometry and the analytic geometry of projective varieties over $\mathbb{C}$ are the same.

Notice that theorem 1 does not imply that the morphism $H^q(X, \mathcal{F}) \to H^q(X, i^*\mathcal{F})$ is an isomorphism. Indeed, if we consider the constant sheaf of rational functions over X, we have $H^q(X, \mathcal{K}_X) = 0$ and $H^q(X^h, \mathcal{K}_X)$ given by spaces with dimension equal to the Betti numbers of X^h. Even if $\mathcal{K}_X$ is not coherent, it is union of coherent sheaves and thus we would have an isomorphism if $H^q(X, \mathcal{F}) \to H^q(X, i^*\mathcal{F})$ was an isomorphism for $\mathcal{F}$ coherent.

4.1 Proof of Theorem 4.1

The proof is based on reducing the theorem to the structural sheaf over $\mathbb{P}^n$. In order to perform this, we need some results:

Lemma 4.4. *Suppose $X \subset \mathbb{P}^n$. Let $\mathcal{F}$ be a sheaf over X and let $\mathcal{F}^{ext}$ be its extension by zeros to $\mathbb{P}^n$. Then we have:*

$$H^q(X, \mathcal{F}) = H^q\left(\mathbb{P}^n, \mathcal{F}^{ext}\right) \quad \text{and} \quad H^q\left(X^h, \mathcal{F}^h\right) = H^q\left(\mathbb{P}^n, (\mathcal{F}^{ext})^h\right).$$

Proof. We know that $\mathcal{F}^{ext}(U) = \mathcal{F}(U \cap X)$ for any $U \subset \mathbb{P}^n$. So, for any covering $\mathscr{U}$ of $\mathbb{P}^n$, we get a covering $\mathscr{U}'$ of X via intersection. The definition o $\mathcal{F}^{ext}$ will give us bijection:

$$C^q(\mathscr{U}, \mathcal{F}^{ext}) = C^q(\mathscr{U}', \mathcal{F}),$$

that is well defined in the cohomology.

The well defineteness on the cohomology follows from the fact that this map commutes with the boundary operator (since the operations with sections on $\mathcal{F}^{ext}$ and $\mathcal{F}$ are the same) and commutes with the colimit, since if $\mathscr{U} \leq \mathscr{V} \implies \mathscr{U}' \leq \mathscr{V}'$. The map of course commutes with restrictions, by definition. We get a morphism:

$$H^q(\mathbb{P}^n, \mathcal{F}^{ext}) = H^q(X, \mathcal{F}).$$

Notice that this prove does not use the fact that $\mathcal{F}$ is coherent, algebraic or that we are in $\mathbb{P}^n$. It is valid for any sheaf defined on a closed subset of any topological space. □

From now on, we assume $X = \mathbb{P}^n$.

Lemma 4.5. *The theorem is valid for the sheaf $\mathcal{O}$.*

Proof. In order to compute the cohomologies of the projective space, we can recall basic facts. The cohomology of $\mathcal{O}(r)$ is zero for all $q > 0$ except for n, in which it has dimension $\binom{-r-1}{n}$ So, for $r = 0$, we have that all cohomologies are zero, excecpt for the 0th cohomology, which is $\mathbb{C}$.

In the other hand, the cohomology with coefficients $\mathcal{H}$ can be computed via Dolbeaut cohomology, as we have a resolution:

$$0 \to \mathcal{H} \to \Omega^{0,0} \xrightarrow{\bar{\partial}} \Omega^{0,1} \xrightarrow{\bar{\partial}} \Omega^{0,2} \xrightarrow{\bar{\partial}} \dots .$$

Since the Dolbeaut cohomology of the projective space is zero for any $q > 0$, we have that all cohomologies are zero except for the first, which is given by $\mathbb{C}$.

This shows the cohomology groups are isomorphic (are all zero). □

Lemma 4.6. *The theorem is valid for the sheaves $\mathcal{O}(r)$.*

Proof. We proceed by induction on $n = \dim X$. Take any hyperplane E on X given by a linear homogenous polynomial t. We get an exact sequence:

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_E \to 0,$$

where $\mathcal{O}_E$ is the extension by zeros of the structural sheaf of E and the first map is the multiplication by t. As $\mathcal{O}(r)$ is locally free, tensoring by it preserves exactness. We have:

$$0 \to \mathcal{O}(r-1) \to \mathcal{O}(r) \to \mathcal{O}_E(r) \to 0.$$

As E is isomorphic to $\mathbb{P}^{n-1}$, we can apply our induction hypothesis. This implies that, in the commutative diagram induced by long exact sequences

$$\begin{array}{ccccccccccc} \dots \to & H^{q-1}(X,\mathcal{O}_E) & \to & H^q(X,\mathcal{O}(r-1)) & \to & H^q(X,\mathcal{O}(r)) & \to & H^q(X,\mathcal{O}_E(r)) & \to & H^{q+1}(X,\mathcal{O}(r-1)) & \to \dots \\ & \varepsilon\downarrow & & \varepsilon\downarrow & & \varepsilon\downarrow & & \varepsilon\downarrow & & \varepsilon\downarrow & \\ \dots \to & H^{q-1}(X^h,\mathcal{O}_E^h) & \to & H^q(X^h,\mathcal{O}(r-1)^h) & \to & H^q(X^h,\mathcal{O}(r)^h) & \to & H^q(X^h,\mathcal{O}_E(r)^h) & \to & H^{q+1}(X^h,\mathcal{O}(r-1)^h) & \to \dots \end{array},$$

we have that the first and fourth vertical arrows are isomorphisms. By the five lemma, if the fact is true for $\mathcal{O}(r-1)$, it is true for $\mathcal{O}(r)$. We end the proof by observing that, by Lemma 4.5, the thorem is valid for $r = 0$, and thus it is true for any r. □

Lemma 4.7. *Let $\mathcal{F}$ be an algebraic coherent sheaf over X. Then there exists an exact sequence*

$$0 \to \mathcal{R} \to \mathcal{L} \to \mathcal{F} \to 0$$

of algebraic coherent sheaves with $\mathcal{L}$ being isomorphic to $\mathcal{O}(r)^k$ for some k and r.

Now, to finish the proof, we proceed by inverse induction on q. If $q > 2r$, all the cohomology groups are zero, and therefore there is nothing to prove. Now, by 4.7, we can find an exact sequence

$$0 \to \mathcal{R} \to \mathcal{L} \to \mathcal{F} \to 0$$

for which $\mathcal{L}$ is a direct sum of $\mathcal{O}(n)$. By passing to the long exact sequence:

$$\begin{array}{ccccccccccc} \dots \to & H^q(X,\mathcal{R}) & \to & H^q(X,\mathcal{L}) & \to & H^q(X,\mathcal{F}) & \to & H^{q+1}(X,\mathcal{R}) & \to & H^{q+1}(X,\mathcal{L}) & \to \dots \\ & \varepsilon_1\downarrow & & \varepsilon_2\downarrow & & \varepsilon_3\downarrow & & \varepsilon_4\downarrow & & \varepsilon_5\downarrow & \\ \dots \to & H^q(X^h,\mathcal{R}^h) & \to & H^q(X^h,\mathcal{L}^h) & \to & H^q(X^h,\mathcal{F}^h) & \to & H^{q+1}(X^h,\mathcal{R}^h) & \to & H^{q+1}(X^h,\mathcal{L}^h) & \to \dots \end{array},$$

To end the proof, we apply the five lemma two times. First, we use that ε_5 and ε_2 are isomorphisms by the fact that $\mathcal{L}$ is of the form $\mathcal{O}(r)^k$ and by Lemma 4.6. By the induction hypothesis, ε_4 is also an isomorphism. By the five lemma, ε_3 is sujective. Applying the same argument for $\mathcal{R}$, we can assume ε_1 is surjective. Now we apply the five lemma again to conclude that ε_3 is injective. This shows ε_3 is an isomorphism and we finish the proof.

4.2 Proof of Theorem 4.2

To prove 4.2, we need to define a morphism between the analytification of the sheaf of morphisms $\mathrm{Hom}(\mathcal{F},\mathcal{G})^h$ and the analytic sheaf $\mathrm{Hom}(\mathcal{F}^h,\mathcal{G}^h)$.

Each $\varphi \in \mathrm{Hom}(\mathcal{F},\mathcal{G})_x$ defines a morphism $\varphi(\mathcal{F}^h,\mathcal{G}^h)_x$ for any point $x \in X$. So, we get a map

$$m : i^*\mathrm{Hom}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}(\mathcal{F}^h,\mathcal{G}^h),$$

which extends to a map:

$$m : \mathrm{Hom}(\mathcal{F},\mathcal{G})^h \to \mathrm{Hom}(\mathcal{F}^h,\mathcal{G}^h).$$

Lemma 4.8. *The morphism m is an isomorphism.*

Proof. As $\mathcal{F}$ is coherent, we have that the sheaf Hom and the stalks commute, in the sense that

$$\mathrm{Hom}(\mathcal{F},\mathcal{G})_x = \mathrm{Hom}(\mathcal{F}_x,\mathcal{G}_x).$$

This can be seen using the fact coherent sheaves are locally genereated by sections. By the fact that $\mathcal{F}^h$ is also coherent, we get

$$\mathrm{Hom}(\mathcal{F}^h,\mathcal{G}^h)_x = \mathrm{Hom}(\mathcal{F}_x \otimes \mathcal{H}_x, \mathcal{G}_x \otimes \mathcal{H}_x)$$

The lemma, therefore, reduces to show that the natural map

$$m_x : \mathrm{Hom}(\mathcal{F}_x,\mathcal{G}_x) \otimes H_x \to \mathrm{Hom}(\mathcal{F}_x \otimes \mathcal{H}_x, \mathcal{G}_x \otimes \mathcal{H}_x)$$

is an isomorphism.

Indeed, as $\mathcal{H}_x$ is $\mathcal{O}_x$-flat (3.7), we have that m_x is an isomorphism. This is purely algebraic. If E and F are A-modules, with E of finite type, and B is A-flat, we have that $\mathrm{Hom}_A(E,F) \otimes_A B \cong \mathrm{Hom}_B(E \otimes_A B, F \otimes_A B)$. Indeed, this follows by writing the sequence $L_1 \to L_0 \to E \to 0$, that exists because E is finitely generated. We have the following exact sequences:

$$0 \to \mathrm{Hom}(E,F) \otimes B \to \mathrm{Hom}(L_0,F) \otimes B \to \mathrm{Hom}(L_1,F) \otimes B$$

(by applying Hom, which is left exact and then the functor $\otimes$, which is exact for A-flat modules), and

$$0 \to \mathrm{Hom}(E \otimes B, F \otimes B) \to \mathrm{Hom}(L_0 \otimes B, F \otimes B) \to \mathrm{Hom}(L_1 \otimes B, F \otimes B).$$

As the fact is trivial for free modules, we get the fact for E.

Applying this to $\mathcal{F}_x$, we conclude the proof.

□

Now we can finish the proof. Denote $\mathcal{A} = \mathrm{Hom}(\mathcal{F},\mathcal{G})$ and $\mathcal{B} = \mathrm{Hom}(\mathcal{F}^h,\mathcal{G}^h)$. Consider the morphism $\varepsilon : H^0(X,\mathcal{A}) \to H^0(X^h,\mathcal{A}^h) \cong H^0(X^h,\mathcal{B}^h)$, where the last isomorphism is given by m.

Theorem 4.2 consists in the assertion that ε is an isomorphism. Although, theorem 4.1 (which can be applied since $\mathcal{A}$ is coherent) says exactly that.

4.3 Proof of Theorem 4.3

In order to prove 4.3, we need two important results which are versions of Cartan's A and B theorems. It is not trivial to obtain this versions from the usual ones stated for Stein manifolds. Although, the idea is the same as the one from algebraic geometry, where one uses the fact that coherent sheaves have vanishing cohomology over affine varieties.

Theorem 4.9 (Cartan's A Theorem). *Let $\mathcal{M}$ be an analytic coherent sheaf over $\mathbb{P}^r$. Then there exists an integer n_0 such that for $n \geq n_0$ and for all $x \in \mathbb{P}^r$, the module $\mathcal{M}(n)_x$ is generated by global sections, that is, by elements from $H^0(\mathbb{P}^r, \mathcal{M}(n))$.*

Theorem 4.10 (Cartan's B Theorem). *Let E be an hyperplane of $\mathbb{P}^r$ and let $\mathcal{A}$ be an analytic coherent sheaf over E. Then there exists an n_0 such that, for $n \geq n_0$ we have $H^q(E^h, \mathcal{A}(n)) = 0$ for all $q > 0$.*

With this two results in hands, we can proceed to the proof 4.3. First, we adress the uniqueness of the sheaf $\mathcal{F}$. Suppose we had another sheaf $\mathcal{G}$ such that $\mathcal{F}^h \cong \mathcal{G}^h$ by a morphism γ. From 4.2, we have that γ is induced by a unique morphism $\varphi : \mathcal{F} \to \mathcal{G}$. We can write the exact sequence

$$0 \to \mathcal{A} \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \to \mathcal{B} \to 0$$

where $\mathcal{A}$ is the kernel of φ and $\mathcal{B}$ is its cokernel.

By 3.13a, we can pass to the analytifications and get the exact sequence

$$0 \to \mathcal{A}^h \to \mathcal{F}^h \xrightarrow{\gamma} \mathcal{G}^h \to \mathcal{B}^h \to 0.$$

As γ is isomorphism, we conclude $\mathcal{A}^h = \mathcal{B}^h = 0$. By 3.13b, we conclude that $\mathcal{A}$ and $\mathcal{B}$ are also 0. This implies φ is an isomorphism and show the uniqueness.

For the existence, we first prove that X can be taken to be $\mathbb{P}^r$. Indeed, suppose $\mathcal{M}$ is an analytic sheaf on $Y^h \subset \mathbb{P}^r$. Extending by zeros, we get a sheaf $\tilde{\mathcal{M}}$ on $\mathbb{P}^r$. Now, if we assume that we have the result for $\mathbb{P}^r$, we can find a sheaf $\mathcal{G}$ such that $\mathcal{G}^h = \tilde{\mathcal{M}}$. Now, restricting this sheaf to Y (which is possible since the sheaf $\mathcal{G}$ has stalks zero outside Y, we get a sheaf $\mathcal{F}$ for which $\tilde{\mathcal{F}} = \mathcal{G}$. By 3.14, we conclude that $\mathcal{F}^h = M$ as we wanted to show.

So assume $X = \mathbb{P}^r$. By 4.9, we have that $\mathcal{M}(n)$ is a quotient of the free sheaf $\mathcal{H}^p$ for some p. After tensoring, we obtatin that $\mathcal{M}$ is a quotient of $\mathcal{H}(-n)^p$. If we denote by $\mathscr{L}_0$ the sheaf $\mathcal{O}(-n)^p$, we have an exact sequence:

$$0 \to \mathcal{R} \to \mathscr{L}_0^h \to \mathcal{M} \to 0$$

By the applying the same argument for $\mathcal{R}$, we obtain another algebraic sheaf $\mathscr{L}_1$ of the form $\mathcal{O}(-m)^k$ and an exact sequence

$$\mathscr{L}_1^h \to \mathscr{L}_0^h \to \mathcal{M} \to 0. \qquad (*)$$

By 4.2, the first map of the sequece above is induced by a morphism $f : \mathscr{L}_1 \to \mathscr{L}_0$ between algebraic sheaves. Now, we define $\mathcal{F} := \mathrm{coker}(f)$.

This yelds a sequence

$$\mathscr{L}_1 \to \mathscr{L}_0 \to \mathcal{F} \to 0.$$

As the analytification is an exact functor (3.13a), we obtain a sequence

$$\mathscr{L}_1^h \to \mathscr{L}_0^h \to \mathcal{F}^h \to 0,$$

which, togheter with $(*)$ shows that $\mathcal{F}^h = M$, as desired.

5 Applications: Chow's Theorem and Betti Numbers

In this section we show two important applications of GAGA theorems. The Chow's theorem, which state that any analytic subvariety of $\mathbb{P}^n$ is in fact a projective algebraic variety and we also show that Betti numbers, which are topological invariants of analytic varieties, have an algebraic character. This last result can be applied to a conjecture of Weil about varieties over fields of algebraic numbers.

5.1 Betti Numbers

Before Serre's article, Weil conjectured the following:

Theorem 5.1. *Let V be a smooth projective algebraic varieity over a field of algebraic numbers K. All complex variety X obtained from V by extending K to $\mathbb{C}$ have the same Betti Numbers, no matter what extension is chosen.*

The content of this result is actually the fact that the topology of the variety over $\mathbb{C}$ is strongly restricted by the algebraic structure over K.

We now state a lemma which will be essencial to prove 5.1.

Lemma 5.2. *Let $\sigma : \mathbb{C} \to \mathbb{C}$ be a field automorphism. σ obviously induces a morphism $\mathbb{P}^r \to \mathbb{P}^r$ via $[t_0 : \cdots : t_r] \mapsto [\sigma(t_0) : \cdots : \sigma(t_r)]$. Let $X \subset \mathbb{P}^r$ be smooth a projective variety and X^σ be the image of X by the morphism induced by σ. We have that X^σ is smooth and $b_k(X) = b_k(X^\sigma)$.*

Proof. First, notice that X^σ is smooth by the Jacobian criteria. Indeed, if $f \in I(X)$, we have that $f \circ \sigma^{-1} \in I(X^\sigma)$. As σ^{-1} is an automorphism, it will take $\det(\mathrm{Jac}(X))$ to $\det(\mathrm{Jac}(X^\sigma))$ and if one is non zero the other is non zero also.

For the second part, we consider the analytifications of X and X^σ and use the Hodge decomposition to write:

$$b_k(X) = \sum_{p+q=k} h^{p,q}(X) \qquad \text{and} \qquad b_k(X^\sigma) = \sum_{p+q=k} h^{p,q}(X^\sigma)$$

where $h^{p,q}(X) = \dim H^q(X^h, \Omega^p(X)^h)$ and the same for X^σ.

By 4.1, we can write $h^{p,q}(X) = \dim H^q(X, \Omega^p(X))$, where $\Omega^p(X)$ is the sheaf of algebraic differential forms of degree p.

In the algebraic context, it is easy to see that any regular algebraic form ω on an Z-open set of X induces a regular algebraic form ω^σ on a Z-open set on X^σ. This induces an isomorphism on the Cech complex that goes to the cohomology. Therefore, we get that $H^q(X, \Omega^p) \cong H^q(X^\sigma, \Omega^p)$ and, therefore, $h^{p,q}(X) = h^{p,q}(X^\sigma)$. By the formulas above, the Betti numbers are the same. □

The proof of the theorem is now a simple consequence of this lemma:

Proof of 5.1. Consider two extensions $K \to \mathbb{C}$. They differ by an automorphism of $\mathbb{C}$. In other words, the two varieties obtained are of the form X and X^σ for some σ.

Now, lemma 5.2 finishes the proof. □

5.2 Chow's Theorem

Another very important application of GAGA is the famous Chow's Theorem. It can be stated as below:

Theorem 5.3. *Let $X \subset \mathbb{P}^n$ be a closed analytic subvariety. Then X is algebraic.*

This surprising result was first proven by Chow in his paper [**?**] from 1949, using more explicit methods. It's equally surprising that, via GAGA, it is almost trivial:

Proof. Consider the sheaf $\mathcal{H}_X$, the structural sheaf of X, whose extension by zeroes is coherent over $\mathbb{P}^n$. By 4.3, there exists an algebraic sheaf $\mathcal{F}$ such that $F^h = \mathcal{H}_X$. Now, by 3.13b, we have that the supports of $\mathcal{F}$ and F^h are the same. But we know that the support of an algebraic coherent sheaf is always Z-closed. That means that X is Z-closed and it finishes the proof. □

This can be extended to any algebraic variety:

Theorem 5.4. *If X is an algebraic variety, any compact analytic subvariety of X is algebraic.*

We present a nice application of these results:

Theorem 5.5. *Let $f : X \to Y$ be a holomorphic map between a algebraic varieties with X compact. Then f is a regular morphism.*

Proof. Let T be the graph of f. As X is compact and T can be realized as image of X, T is also compact. As f is holomorphic, T is an analytic subvariety of $X \times Y$. By 5.4, we conclude that T is algebraic. But this implies that f is regular by 3.11. □

6 Final Considerations

We hope that this text is able to give the reader at least the main ideas on why GAGA is true and the main techniques Serre used on his paper.

Another application Serre gives include the relationship between algebraic and analytic fibrations. Also, we omitted some proofs, speacially for known facts in order to make the text more concise.

This results can be generalized to the language of schemes, which was not avaible to Serre when he wrote his paper. For details, it may be useful to look at [Grothendieck, 1985].

Although we hope our text is understandable, we strongly recommend the reader to take a look at Serre's [Serre, 1956]. That paper is almost completely self contained and all the proofs are very clear.

Enjoy GAGA!

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