



Cohomological
Field
Theories
from
Gauged
Linear
Sigma
Models

Based
on
joint work
with B. Kim

Dedicated to
the memory
of B. Kim

I : GLSMs

II : CohFT data for GLSMs

III : Results

IV : Construction

I GLSMs

Data: $(V, \Gamma, \chi, \omega, r)$

- V \mathbb{C} -vector space
- $\Gamma \subseteq \text{GL}(V)$ lin. red. gp
- $\chi : \Gamma \rightarrow \mathbb{C}^\times$
- V a \mathbb{Q} -character
- $\omega \in ((\text{Sym } V^*) \otimes \mathbb{C}_\chi)^G$
- $G := \ker \chi$
- $\theta := \gamma|_G$

Require

- $V^{ss}(\gamma) = V^{ss}(\theta) = V^s(\theta)$
- $Z(d\omega)$ is proper

Roughly:

$$\mathbb{V} \xrightarrow{\cong} G \xrightarrow{\omega} C$$

s.t.

- $\mathbb{V} \xrightarrow{\cong} G$ is a smooth DM-stack
- $\mathbb{Z}(dw)$ is proper

Specialize to

- (Complete intersections) in
 - Projective space
 - Toric varieties
 - Grassmannians
 - Quiver varieties

Goal: Produce an enumerative theory for GLSMs

- Specializes to Gromov-Witten theory of complete intersections in V/G
- Specializes to FJRW theory when G is finite

$$[V/G]$$

$$\xrightarrow{\omega} \mathbb{C}$$

$$\underline{\text{Ex}} \quad \mathbb{C}^x \subset \mathbb{C}^{n+2}$$

$$= \text{Spec } \mathbb{C}[x_0, \dots, x_n, p]$$

weights: 1, ..., 1, -d

$$\text{a) } \theta_+ : \mathbb{C}^x \xrightarrow{\text{id}} \mathbb{C}^x$$

$$\text{b) } \theta_- : \mathbb{C}^x \xrightarrow{i} \mathbb{C}^x$$

$$t \longmapsto t^{-1}$$

- $f(x)$ = homogeneous polynomial of degree d

- $\omega_f = p f(x)$

$$\text{a) } \mathbb{C}^{n+2} /_{\theta_+} \mathbb{C}^x = \text{tot } \mathcal{O}_{P^n}(-d) \xrightarrow{<-, f>} \mathbb{C}$$

Gives GW theory of $Z(f) \subseteq \mathbb{P}^n$

$$\text{b) } \mathbb{C}^{n+2} /_{\theta_-} \mathbb{C}^x = \left[\mathbb{A}^{n+1} /_{2d} \right] \xrightarrow{f} \mathbb{C}$$

Gives FJRW theory

II CohFT data for GLSMs

Defn

A cohomological field theory is:

- \mathcal{H} a graded \mathbb{G} -vector space (state space)
- $\langle - , - \rangle$ a supercommutative pairing
- $1 \in \mathcal{H}$ a distinguished element (unit)
- $\mathcal{D}_{g,r,d} : \mathcal{H}^{\otimes r} \rightarrow H^*(\overline{M}_{g,r})$
(correlators)

- satisfying natural axioms
 - permutation covariance
 - tree
 - loop
 - forgetting tails
 - metric

Ex (GW theory of Z)

Let Z be a smooth variety

$M_{g,r,d}(Z)$ = Moduli of maps

$$C \rightarrow Z$$

• $\mathcal{H} := H^*(Z)$

• $\langle v_1, v_2 \rangle := \sum_{\mathbb{Z}} v_1 \cup v_2$

ev: $M(Z) \xrightarrow{\phi} Z^r$
 $\phi \hookrightarrow (\text{ev}(a_1), \dots, \text{ev}(a_r))$

$$H^*(M_{g,r,d}(Z)) \xrightarrow{\cap [M(Z)]^{vir}} H_*^{vir}(M_{g,r,d}(Z))$$

$\downarrow ev^*$

$$\mathcal{H}^{\otimes r} = H^*(Z^r) \xrightarrow{R_{g,r,d}}$$

$$H_*(\overline{M}_{g,r})$$

|| S.P.D.

$$\Rightarrow H^*(\overline{M}_{g,r})$$

GLSM State Space

$\mathcal{H} := H^*(\mathcal{Q}^*, IV_{\theta/G}, \alpha d\omega)$
 twisted Hodge coh. θ of the
 inertia stack $(H^*(A^*; \partial + \alpha d\omega))$

where

$$IV_{\theta/G} := \coprod_{g \in G/G} V^{ss}(\theta)^g / C_G(g)$$

← conjugacy classes

Kunneth Formula

$$\mathcal{H}^{\otimes r} = H^*(\mathcal{Q}^*, (IV_{\theta/G})^r, \alpha d\omega^r)$$

GLSM Pairing

$$\alpha \in H^*(\mathcal{R}_X^\bullet, d\omega)$$

$$\beta \in H^*(\mathcal{R}_X^\bullet, d\nu)$$

$$\alpha \wedge \beta \in H^*(\mathcal{R}_X^\bullet, d(\omega + \nu))$$

$$X = [V^{ss}(\theta)/G]$$

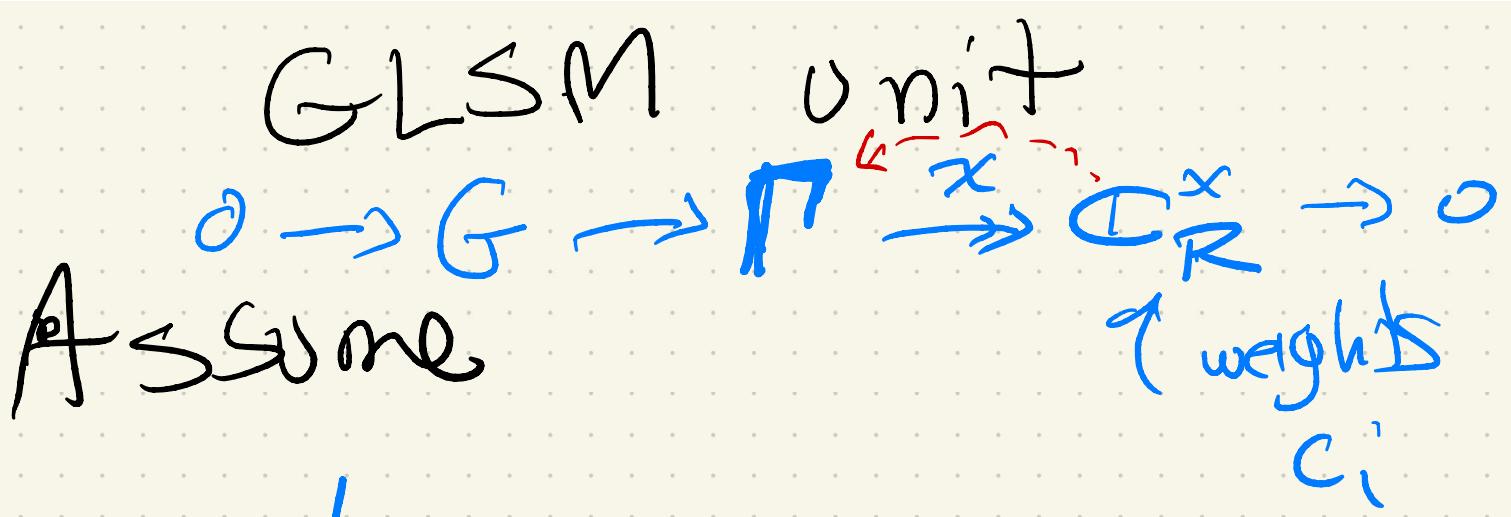
$$\text{inv: } I_X \rightarrow I_X$$

$$\gamma \in (V^{ss})^g \hookrightarrow \gamma \in (V^{ss})^{g^{-1}}$$

$$\text{where } \gamma = e^{\frac{i}{\hbar} \alpha} \hookleftarrow \overset{\text{hom.}}{\deg_0} \underset{\mathbb{C}}{\sim} w$$

$$H^*(\mathcal{R}_{IX}^\bullet, \Lambda d\omega) \cong H^*(\mathcal{R}_{IX}^\bullet, -\Lambda d\omega)$$

$$\langle \alpha, \beta \rangle := \int \alpha \wedge \text{inv}^* \beta$$



$$\textcircled{1} \quad d > c_i \geq 0$$

$$\textcircled{2} \quad \left[V^{ss}(\theta) / G \right] = \left[V^{ss}(\theta) / G \right]^{C_R^X}$$

$$1 := +d \left(\mathcal{R}_{\mathbb{P}}^X / \mathcal{X}^{C_R^X} \right)$$

- $\text{ch}(O \xleftarrow{\quad} i^* O_{\mathbb{P}^{C_R^X}})$

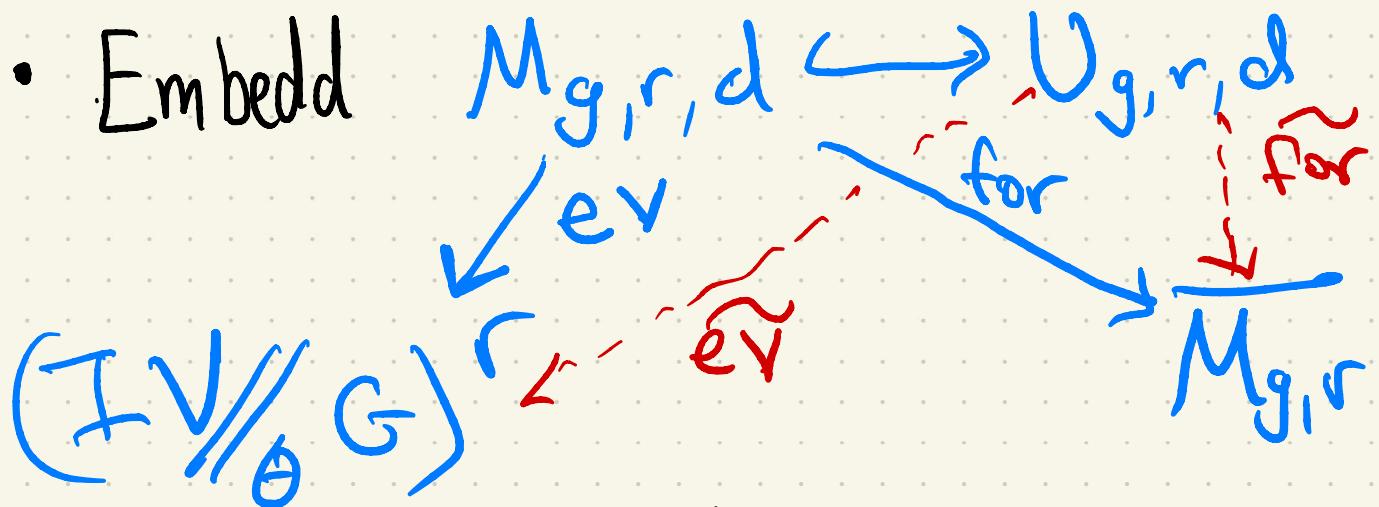
$H^*(\mathcal{R}_{\mathbb{P}}^X, \wedge \omega)$

where

$$\mathcal{J} := G \cap C_R^X$$

GLSM Correlators (Idea)

$M_{g,r,d} = M_{g,r,d}^{LG}(V, G, \theta, w)$ be the moduli space of LG quasi maps to $\mathcal{C} \rightarrow Z(\partial w) \subseteq V//_G$



- Construct a virtual cycle in twisted Hodge cohomology of $U_{g,r,d}$

$[M]_{g,r,d}^{\text{vir}} \in H^*_{M_{g,r,d}}(R_{U_{g,r,d}}, -d(\tilde{ev}^* w^{\text{vir}}))$ supported on $M_{g,r,d}$ (which is proper by FJR 2018)

GLSM Correlators

$$\Lambda[M]_{g,r,d}^{\text{vir}}$$

$$H^*(U_{g,r,d}, \Lambda^d(\tilde{ev}_0 \omega^{\otimes r})) \rightarrow H^*_{M_{g,r,d}}(U_{g,r,d})$$

$$\xrightarrow{\text{ex}^*}$$

$$\downarrow \quad \text{for } *$$

$$H^*(R_{IV//G^r}, \Lambda^d \omega^{\otimes r}) \dashrightarrow H^*(\bar{M}_{g,r})$$

$\xrightarrow{\text{R}_{g,r,d}}$

$$H^{\otimes r}$$

III : Results

- Fan-Jarvis-Ryan (2013)
- Polishchuk-Vaintrob (2016)

G finite

Purely alg. version
where the virtual
class comes a
matrix factorization

- Kiem-Li (2018)

Using cosection
(localization)

- Fan-Jarvis-Ryan (2018)

"Narrow sectors"
for general GLSMs

- Gican-Fontanine-F-Guére -
Kim-Shoemaker (2018)

Convex hybrid
models
(all sectors)

- F-Kim (2020)

General case

Thm (CF-F-G-K= S, 2018)

Enumerative Invariants for
convex hybrid models specialize
to FJRW theory and

Gromov-Witten using
the cosection (localized virtual
cycle).

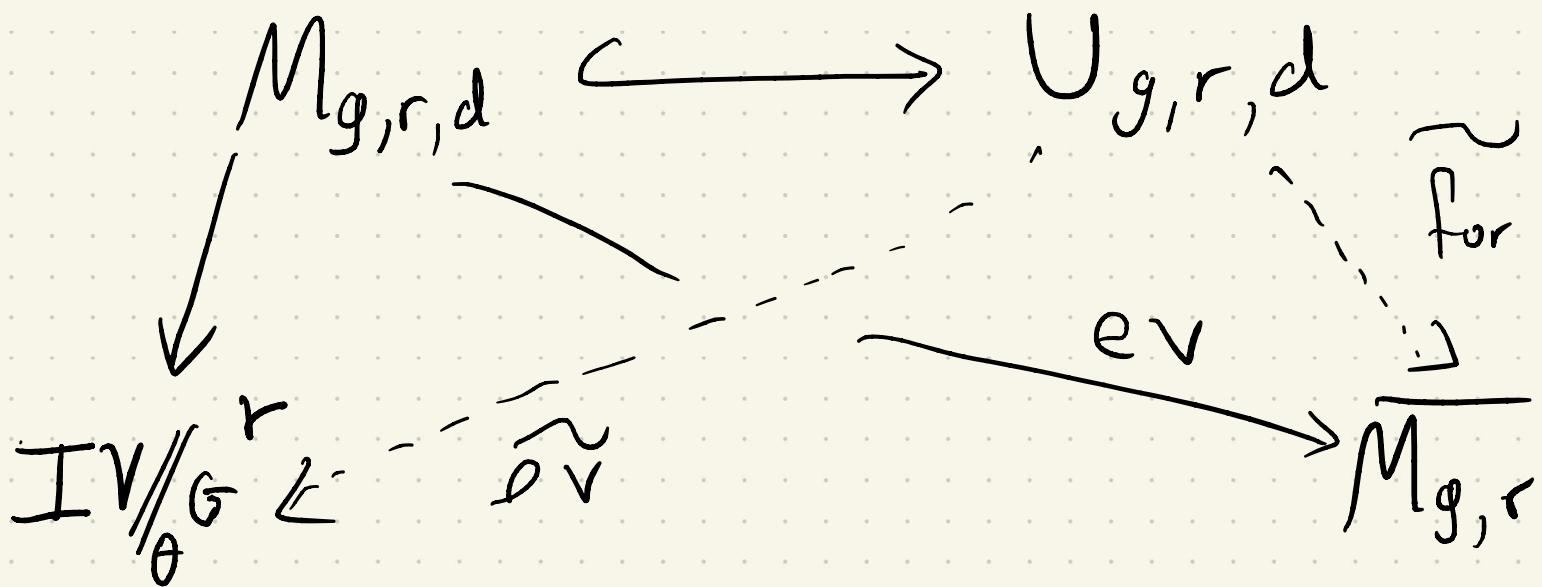
Thm (Kim-Oh, 2018) The cosection
(localized virtual) agrees with
the Behrend-Fantechi virtual cycle
up to sign.

Thm (F-Kim, 2020) The general
GLSM invariants form a
CohFT.

IV

Construction

- Embed $M_{g,r,d}$ in a smooth space



- Find a "virtual" matrix factorization $K_{g,r,d}$ on $(U_{g,r,d}, \widehat{ev}^\circ w^{\boxplus r})$ supported on $M_{g,r,d}$
- $[M]_{g,r,d}^{vir} := \text{fd}(B_{g,r,d}) \text{ ch } (K_{g,r,d})$

Construction of $K_{g,r,d}$,

LG quasimap data,

- C genus g curve
- $q = (q_1, \dots, q_r)$ marked points
- P principal \mathbb{P} -bundle
- $\chi: P \times_{\mathbb{P}} C \rightarrow W_C^{\log}$
- $u: C \rightarrow P \times_{\mathbb{P}} V$

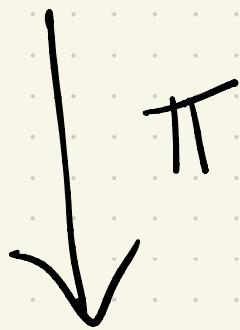
$$\mathcal{F} \times_{\mathcal{G}} V$$

vector
bundle
on
 \mathcal{C}



$$\mathcal{C}$$

universal
curve



$$\{(c, g, P, K)\}$$

$$R\pi_*(\mathcal{F} \times_{\mathcal{G}} V) = [A \xrightarrow{d} B]$$

$$R\pi_*(\mathcal{F} \times_{\mathcal{F}} V) = [A \xrightarrow{d} B]$$

$$\ker d = H^0(\mathcal{F} \times_{\mathcal{F}} V / G)$$

extra
data

want to
construct a
co-section
using ω

$$\text{tot } P^* B$$

$$Z(B) \subseteq V_{g,r,d} \subseteq \text{tot } A$$

$$\downarrow P$$

$$Z(C, q, P, K)$$

$$Z(\alpha) = M_{g,r,d}$$

$$K \approx [\wedge^{\text{even}} P^* B^V \leftarrow \wedge^{\text{odd}} P^* B^V]$$

$$\partial = \wedge B + \wedge \alpha$$

$$\text{Supp}(K) = Z(\alpha)$$

$$= M_{g,r,d}$$

Problem: In general α only exists locally on $U_{g,r,d}$

$$\alpha \in H^1(U_{g,r,d}, \mathcal{K}(B))$$

Want: To find a matrix factorization which is locally the Koszul factorization

Observation

Koszul factorizations come from sheaves of CDGAs:

- \mathcal{X} DM stack
- $\alpha(A, d)$ sheaf of CDGAs over \mathcal{X}
- $w \in \Gamma(X, \mathcal{O}_X)$
- $\alpha \in \Gamma(X, A_{-1})$ s.t. $d(\alpha) = w$

Then

$$A^{\text{even}} \xrightarrow{\partial} A^{\text{odd}}$$
$$\xleftarrow{\partial} A^{\text{odd}}$$

$$\partial = d + \bullet \alpha$$

graded Leibnitz rule $\Rightarrow \partial^2 = w$

\therefore this is a matrix factorization

Idea:

To realize

$\text{det} H^*(\text{tot } A, \mathcal{K}(B))$

we need to replace $\mathcal{K}(B)$

by a Γ -acyclic complex

but retain the CDGA structure

$\mathcal{K}(B) \xrightarrow{\text{Godement}} G^* \mathcal{K}(B) \xrightarrow{\text{Thom-Sullivan}} \text{Th}^* G^* \mathcal{K}(B)$

Γ -acyclic
cosimplicial
sheet of
CDGAs

Γ -acyclic
sheet
of
CDGAs

Then

- $\alpha \in \text{Th}^G K(B)$

- and $d(\alpha) = w$

$$K_{gr,rd} := \left[(\text{Th}^G K(B))^{\text{even}} \xrightarrow{\quad} (\text{Th}^G K)^{\text{odd}} \right]$$

locally looks like

$$\left[\wedge^{\text{even}} p^* \beta^\vee \xrightarrow{\quad} \wedge^{\text{odd}} p^* \beta^\vee \right]$$