Cohomological Field Theories from Gauged Linear Sigma Models

Based on joint work with B. Kim

Dedicated to the memory of B. Kim

I : GLSMs

II : CohFT data for GLSMs

III : Results

IV : Construction

I GLSMs

Data: $(V, \Gamma, \chi, w, \nu)$

- V $\mathbb{C}$-vector space
- $\Gamma \leq Gl(V)$ lin. red. gp
- $\chi : \Gamma \twoheadrightarrow \mathbb{C}^\times$
- ν a $\mathbb{Q}$-character
- $w \in ((\operatorname{Sym} V^\vee) \otimes \mathbb{C}_\chi)^G$
- $G := \ker \chi$
- $\theta := \nu|_G$

Require

- $V^{ss}(\nu) = V^{ss}(\theta) = V^s(\theta)$
- $Z(dw)$ is proper

Roughly:

$$V/\!\!/_\theta G \xrightarrow{w} \mathbb{C}$$

s.t.

- $V/\!\!/_\theta G$ is a smooth DM-stack
- $Z(dw)$ is proper

Specialize to

- (Complete intersections) in
  Projective space
  toric varieties
  Grassmannians
  Quiver Varieties
  ...

<u>Goal</u>: Produce an enumerative theory for GLSMs

- specializes to Gromov-Witten theory of complete intersections in $V /\!/_\theta G$
- Specializes to FJRW theory when G is finite

$$[V/G] \xrightarrow{w} \mathbb{C}$$

Ex $\mathbb{C}^\times \circlearrowright \mathbb{C}^{n+2}$

$= \operatorname{Spec} \mathbb{C}[x_0, \dots, x_n, p]$

weights: $1, \dots, 1, -d$

a) $\theta_+ : \mathbb{C}^\times \xrightarrow{id} \mathbb{C}^\times$

b) $\theta_- : \mathbb{C}^\times \xrightarrow{i} \mathbb{C}^\times$

$t \longmapsto t^{-1}$

- $f(x)$ = homogeneous polynomial of degree d
- $w := p f(x)$

a) $\mathbb{C}^{n+2} /\!\!/_{\theta+} \mathbb{C}^\times = \operatorname{tot} \mathcal{O}_{\mathbb{P}^n}(-d) \xrightarrow{\langle -, f \rangle} \mathbb{C}$

Gives GW theory of $Z(f) \subseteq \mathbb{P}^n$

b) $\mathbb{C}^{n+2} /\!\!/_{\theta-} \mathbb{C}^\times = [\mathbb{A}^{n+1} / \mathbb{Z}_d] \xrightarrow{f} \mathbb{C}$

Gives FJRW theory

II CohFT data for GLSMs

Defn

A cohomological field theory is:

- $\mathcal{H}$ a graded $\mathbb{C}$-vector space (state space)
- $\langle -,- \rangle$ a supercommutative pairing
- $1 \in \mathcal{H}$ a distinguished element (unit)
- $\Omega_{g,r,d} : \mathcal{H}^{\otimes r} \longrightarrow H^*(\overline{M}_{g,r})$ (correlators)

- satisfying natural axioms
  - permutation covariance
  - tree
  - loop
  - forgetting tails
  - metric

Ex (GW theory of Z)

Let Z be a smooth variety

$M_{g,r,d}(Z)$ = Moduli of maps $\mathcal{C} \to Z$

- $\mathcal{H} := H^*(Z)$
- $\langle v_1, v_2 \rangle := \int_Z v_1 \cup v_2$

$ev : M(Z) \to Z^r$
$\varphi \mapsto (\varphi(a_1), \dots, \varphi(a_r))$

$$\begin{tikzcd}
H^*(M_{g,r,d}(Z)) \arrow[r, "\cap [M(Z)]^{vir}"] & H_*(M_{g,r,d}(Z)) \arrow[d, "for_*"] \\
\mathcal{H}^{\otimes r} = H^*(Z^r) \arrow[u, leftarrow, "ev^*"] \arrow[dr, dashed, "\Omega_{g,r,d}"] & H_*(\overline{M}_{g,r}) \\
 & H^*(\overline{M}_{g,r}) \arrow[u, "\text{P.D.}", "\text{\rotatebox{90}{$\cong$}}"']
\end{tikzcd}$$

GLSM State Space

$$\mathcal{H} := \mathbb{H}^*(\Omega^\bullet_{IV/\!/_\theta G}, \wedge dw)$$

twisted Hodge coh. of the inertia stack $(H^*(A^\bullet, \partial + \wedge dw))$

where

$$IV/\!/_\theta G := \coprod_{g \in G/G} \left[V^{ss}(\theta)^g / C_G(g) \right]$$

← conjugacy classes

Kunneth Formula

$$\mathcal{H}^{\otimes r} = \mathbb{H}^*(\Omega^\bullet_{(IV/\!/_\theta G)^r}, \wedge dw^{\boxplus r})$$

GLSM Pairing

$\alpha \in \mathbb{H}^*(\Omega^\bullet_{\mathcal{X}}, dw)$

$\beta \in \mathbb{H}^*(\Omega^\bullet_{\mathcal{X}}, dv)$

$\alpha \wedge \beta \in \mathbb{H}^*(\Omega^\bullet_{\mathcal{X}}, d(w+v))$

$\mathcal{X} = [V^{ss}(\theta)/G]$

$\mathrm{inv}: I\mathcal{X} \longrightarrow I\mathcal{X}$

$x \in (V^{ss})^g \longmapsto \zeta x \in (V^{ss})^{g^{-1}}$

where $\zeta = e^{\frac{\pi i}{d}}$ ← homo. deg of w

$\mathbb{H}^*(\Omega^\bullet_{I\mathcal{X}}, \Lambda dw) \cong \mathbb{H}^*(\Omega^\bullet_{I\mathcal{X}}, -\Lambda dw)$

$\langle \alpha, \beta \rangle := \int \alpha \wedge \mathrm{inv}^* \beta$

GLSM unit

$$0 \to G \to \Gamma \xrightarrow{\chi} \mathbb{C}^\times_R \to 0$$

↑ weights c_i

Assume

① $d > c_i \geq 0$

② $\left[V^{ss}(\theta)^{\mathbb{C}^\times_R}/G\right] = \left[V^{ss}(\theta)/G\right]^{\mathbb{C}^\times_R}$

$$\mathbb{1} := \mathrm{td}\left(\Omega_{\mathcal{X}^J/\mathcal{X}^{\mathbb{C}^\times_R}}\right)$$

$$\cdot \mathrm{ch}\left(\mathcal{O} \rightleftarrows i_* \mathcal{O}_{\mathcal{X}^{\mathbb{C}^\times_R}}\right)$$

$$\cap$$

$$\mathbb{H}^*(\Omega^\bullet_{I\mathcal{X}}, \wedge dw)$$

where

$$J := G \cap \mathbb{C}^\times_R$$

GLSM Correlators (Idea)

$\mathcal{M}_{g,r,d} = \mathcal{M}^{LG}_{g,r,d}(V, G, \theta, w)$ be the moduli space of LG quasimaps to $\mathcal{C} \longrightarrow Z(\partial w) \subseteq V /\!\!/_\theta G$

- Embedd

$$\begin{tikzcd}
\mathcal{M}_{g,r,d} \arrow[r, hook] \arrow[d, "ev"'] \arrow[dr, "for"] & \mathcal{U}_{g,r,d} \arrow[dl, dashed, "\widetilde{ev}"] \arrow[d, dashed, "\widetilde{for}"] \\
(IV /\!\!/_\theta G)^r & \overline{\mathcal{M}}_{g,r}
\end{tikzcd}$$

- Construct a virtual cycle in twisted Hodge cohomology of $\mathcal{U}_{g,r,d}$

$$[\mathcal{M}]^{vir}_{g,r,d} \in \mathbb{H}^*_{\mathcal{M}_{g,r,d}}\left(\Omega^\bullet_{\mathcal{U}_{g,r,d}}, -d\left(\widetilde{ev}^* w^{\boxplus r}\right)\right)$$

supported on $\mathcal{M}_{g,r,d}$ (which is proper by FJR 2018)

GLSM Correlators

$$
\begin{tikzcd}
\mathbb{H}^*\left(U_{g,r,d}, \wedge d\left(\widetilde{ev} \circ w^{\boxplus r}\right)\right) \arrow[r, "\wedge {[M]}^{vir}_{g,r,d}"] & \mathbb{H}^*_{M_{g,r,d}}\left(U_{g,r,d}\right) \arrow[d, "\widetilde{for}_*"] \\
\mathbb{H}^*\left(\Omega^\bullet_{IV/\!/_\theta G^r}, \wedge d w^{\boxplus r}\right) \arrow[u, "\widetilde{ev}^*"] \arrow[r, dashed, "\Omega_{g,r,d}"'] & \mathbb{H}^*\left(\overline{M}_{g,r}\right)
\end{tikzcd}
$$

$$\rotatebox{90}{$=$}$$

$$\mathcal{H}^{\otimes r}$$

III: Results

- Fan-Jarvis-Ruan (2013) — G finite
- Polishchuk-Vaintrob (2016) — Purely alg. version where the virtual class comes a matrix factorization
- Kiem-Li (2018) — Using cosection localization
- Fan-Jarvis-Ruan (2018) — "Narrow sectors" for general GLSMs
- Ciocan-Fontanine-F-Guéré-Kim-Shoemaker (2018) — Convex Hybrid Models (all sectors)
- F-Kim (2020) — General Case

<u>Thm</u> (CF-F-G-K-S, 2018)
Enumerative Invariants for convex hybrid models specialize to FJRW theory <u>and</u> Gromov-Witten using the cosection localized virtual cycle.

Thm (Kim-Oh, 2018) The cosection localized virtual agrees with the Behrend-Fantechi virtual cycle up to sign.

Thm (F-Kim, 2020) The general GLSM invariants form a CohFT.

IV Construction

- Embedd $M_{g,r,d}$ in a smooth space

$$
\begin{tikzcd}
M_{g,r,d} \arrow[r, hook] \arrow[d] \arrow[dr, "ev"] & U_{g,r,d} \arrow[dl, dashed, "\widetilde{ev}"'] \arrow[d, dashed, "\widetilde{for}"] \\
IV /\!\!/_{\theta} G^r & \overline{M}_{g,r}
\end{tikzcd}
$$

- Find a "virtual" matrix factorization $\mathbb{K}_{g,r,d}$ on $(U_{g,r,d}, \widetilde{ev} \circ w^{\boxplus r})$ supported on $M_{g,r,d}$

- $[M]^{vir}_{g,r,d} := td(B_{g,r,d})\, ch(\mathbb{K}_{g,r,d})$

Construction of $\mathbb{K}_{g,r,d}$

LG quasimap data:

- C genus g curve
- $\underset{\sim}{q} = (q_1, \dots, q_r)$ marked points
- P principal Γ-bundle
- $\varkappa : P \times_\Gamma \mathbb{C}_\chi \longrightarrow \omega_C^{\log}$
- $u : C \longrightarrow P \times_\Gamma V$

$$
\begin{array}{c}
\mathcal{P} \times_{\Gamma} V/G \\
\downarrow \\
\mathcal{C} \\
\downarrow \pi \\
\{(C, q, \mathcal{P}, \kappa)\}
\end{array}
$$

vector bundle on $\mathcal{C}$

universal curve

$$R\pi_*(\mathcal{P} \times_{\Gamma} V/G) = [A \xrightarrow{d} B]$$

$$\mathbb{R}\pi_*\left(\frac{\mathcal{P}\times_\Gamma V}{G}\right) = [A \xrightarrow{d} B]$$

$$\ker d = H^0(\mathcal{P}\times_\Gamma V/G) \leftarrow \text{extra data}$$

$\omega \in$

want to construct a cosection using ω

$$\begin{array}{c} \operatorname{tot} p^*B \\ \downarrow \ \uparrow \\ \operatorname{tot} A \end{array}$$

α

$$Z(\beta) \subseteq U_{g,r,d} \subseteq \operatorname{tot} A$$

$$\operatorname{tot} A \xrightarrow{p} \{(C,q,\mathcal{P},\kappa)\}$$

$Z(\beta) \cong \{(C,q,\mathcal{P},\kappa,u)\}$

$= M^{LG}(V/\!\!/_\theta G)$

$\cong$

$Z(\alpha) = M_{g,r,d}$

$$\mathbb{K} \simeq \left[\Lambda^{\text{even}} p^*B^\vee \underset{\partial}{\overset{\partial}{\rightleftarrows}} \Lambda^{\text{odd}} p^*B^\vee\right]$$

$$\partial = \lrcorner\beta + \wedge\alpha$$

$\operatorname{Supp}(\mathbb{K}) = Z(\alpha,\beta)$

$= M_{g,r,d}$

Problem: In general α only exists locally on $U_{g,r,d}$

$\alpha \in H^1(U_{g,r,d}, \mathcal{K}(B))$

Want: To find a matrix factorization which is locally the Koszul factorization

Observation

Koszul factorizations come from sheaves of CDGAs:

- $\mathcal{X}$ DM stack
- (A, d) sheaf of CDGAs over $\mathcal{O}_{\mathcal{X}}$
- $w \in \Gamma(X, \mathcal{O}_X)$
- $\alpha \in \Gamma(X, A_{-1})$ s.t. $d(\alpha) = w$

Then

$$A^{even} \underset{\partial}{\overset{\partial}{\rightleftarrows}} A^{odd}$$

$$\partial = d + \cdot\alpha$$

graded Leibnitz rule $\Rightarrow \partial^2 = w$

$\therefore$ this is a matrix factorization

Idea:

To realize $\mathcal{L}\mathcal{C}\mathcal{H}^1(\mathrm{tot}\, A, \mathcal{K}(B))$ we need to replace $\mathcal{K}(B)$ by a Γ-acyclic complex but retain the CDGA structure

$$\mathcal{K}(B) \xrightarrow{\text{Godement}} G^{\bullet}\mathcal{K}(B) \xrightarrow{\text{Thom-Sullivan}} Th\, G^{\bullet}\mathcal{K}(B)$$

$G^{\bullet}\mathcal{K}(B)$: Γ-acyclic cosimplicial sheaf of CDGAs

$Th\, G^{\bullet}\mathcal{K}(B)$: Γ-acyclic sheaf of CDGAs

Then

- $\alpha \in \mathrm{Th}^{\bullet} G^{\bullet} K(B)_{-1}$
- and $d(\alpha) = W$

$$\mathbb{K}_{g,r,d} := \left[(\mathrm{Th}^{\bullet} G^{\bullet} K(B))^{even} \rightleftarrows (\mathrm{Th}^{\bullet} G^{\bullet} K(B))^{odd} \right]$$

locally looks like

$$\left[\Lambda^{even} p^{*}B^{\vee} \rightleftarrows \Lambda^{odd} p^{*}B^{\vee} \right]$$