Categories of Line Operators and VOAs

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Motivation

Symmetries

- If a physical system has a group or algebra of symmetries then representations of this group or algebra help in understanding the physics.
- A two-dimensional conformal field theory always has an infinite dimensional Lie algebra, the Virasoro algebra, acting and so it is considered to be a rare example of a quantum field theory that is in principle completely understandable.
- The symmetry algebra of a two-dimensional CFT is a vertex operator algebra (VOA).

Modern Symmetries

- Representations of a group or algebra of a certain type are usually summarized into suitable categories of representations.
- Categories can have more structure, as e.g. a tensor structure with non-trivial commutativity isomorphisms (braidings).
- Complicated categories and additional categorical structures appear more and more in some classes of modern quantum theories.

This Talk

- Additional categorical structures are very hard to study in general.
- I want to give a somehow categorical version of free field realization.
- By a free field realization I vaguely mean a connection of the algebra of interest to a much simpler structure.
- The idea is that the complicated representation theory is completely specified by a nilpotent algebra inside the category of the free field algebra.

Realizations for $\mathfrak{sl}_2$

- The Cartan subalgebra of $\mathfrak{sl}_2$ is just $\mathbb{C}$ and one can view $\mathfrak{sl}_2$ as an algebra in the category of modules of the Cartan subalgebra.
- In this case the nilpotent algebra would just be $\mathbb{C}[e]$ with e the usual nilpotent element of $\mathfrak{sl}_2$ (and this would be inside the universal envelopping algebra).
- The Lie algebra $\mathfrak{sl}_2$ acts on the Lie group SL_2 via left (or right) invariant vector fields and so it is a subalgebra of an algebra of differential operators.

Vertex Operator Algebras

Vertex Operator Algebra (VOA)

A vertex operator (super)algebra $(V, |0\rangle, T, Y)$ is:

- a vector(super) space V (usually over $\mathbb{C}$),
- fields
$$Y(\ \ , z) : V \to \mathrm{End}(V)[[z^{\pm 1}]],$$
- a vacuum $|0\rangle \in V$,
- a translation operator $T \in \mathrm{End}(V)$.
- an action of the Virasoro algebra on V.
- Satisfying various axioms, most importantly locality
$$(z - w)^N [Y(v, z), Y(v', w)] = 0,$$
for all $v, v' \in V$ and sufficiently large $N \in \mathbb{Z}$.
- One writes V for $(V, |0\rangle, T, Y)$.

- There is a similar notion of modules.
- Tensor product is realized by fields $\mathcal{Y}$ that intertwine between modules.
- Associator and commutator of the tensor product is non-trivial as it is defined by analytic continuation of correlation functions of intertwining fields:

$$\left\langle w_4, \mathcal{Y}^2(w_2, z_2)\mathcal{Y}^1(w_1, z_1)w_3 \right\rangle$$

$$\left\langle w_4, \mathcal{Y}^1(w_1, z_1)\mathcal{Y}^2(w_2, z_2)w_3 \right\rangle$$

$$\left\langle w_4, \mathcal{Y}^1\left(\mathcal{Y}^2(w_2, z_2 - z_1)w_1, z_1\right) w_2 \right\rangle$$

- Physics suggests that suitable categories of VOA modules are ribbon.

Ribbon Category

- An abelian category $\mathcal{C}$ with a tensor bifunctor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

  with various properties.
- The tensor product is associative up to natural isomorphisms.
- The tensor product is commutative up to natural isomorphisms (braidings).
- every object has a dual.
- the braiding is balanced.

Historical Highlights

- VOAs are the symmetry algebras of two dimensional conformal field theories (CFTs), 1980's.
- Key ingredient in the proof of monstrous moonshine (Borcherds, Fields medal 1998).
- Axioms of rational CFT lead to modular tensor categories (Verlinde, Moore-Seiberg, 1988; proven by Yi–Zhi Huang, 2005).
- Knot, link and 3-manifold invariants via topological field theories (Witten, Fields medal 1990).
- Ribbon equivalence of categories of affine Lie algebras and quantum groups (Kazhdan-Lusztig 1996)

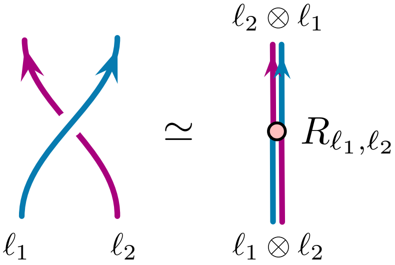
Modern Highlights

- Correspondences of 4-dimensional quantum physics and 2-dimensional CFT (Beem, Rastelli, Gaiotto, ...).
- W-algebras act on moduli spaces of instantons (Vasserot-Schiffmann, Braverman-Finkelberg-Nakajima, ...).
- Quantum geometric Langlands correspondence (Gaitsgory, Frenkel, ...)
- Invariants of 3 and 4–manifolds (Feigin, Gukov, ...)
- Topological recursion (Kontsevich-Soibelman, ...).

Braiding of Line Operators

- Some 3-dimensional $\mathcal{N} = 4$ gauge theories allow for boundary conditions that are **not** topological after twisting. These boundary conditions then support a 2-dimensional VOA (Costello, Gaiotto 2018).
- Categories of line operators should form derived braided tensor categories
- these are in particular non-semisimple and non-finite and often non locally finite categories. In fact they are usually not tame, but wild.

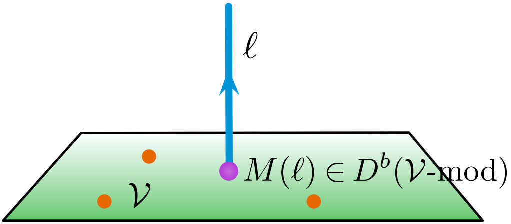
Boundary VOAs



Category of Line Operators

- Question: What do these categories of line operators look like?
- There are various geometric models, but they don't incorporate the braided tensor structure.
- Thus, it is a good idea to use the boundary VOA.

Category of Line Operators and boundary VOA



The boundary VOAs

- The *A*-twisted boundary VOA is defined as a BRST (relative semiinfinite Lie algebra) cohomology of a suitable system of symplectic bosons
- The *B*-twisted boundary VOA is an (unusual) affine vertex superalgebra
- A concrete dictionary is only worked out for abelian theories and in particular in that case boundary VOAs and categories of line operators respect mirror symmetry (Bailin-C-Dimofte-Niu 2023).
- All these boundary VOAs allowed for nice free field realizations and that should be key.

Free field realization

- Let V be a VOA, that embeds conformally into a free field VOA A (e.g. free bosons and/or fermions).
- Let $\mathcal{U}$ be a braided tensor category of V-modules such that A is an object in $\mathcal{U}$.
- Let $\mathcal{C}$ be the braided tensor category of A-modules that lie in $\mathcal{U}$.
- $\mathcal{C}$ is just the category of vector spaces graded by some abelian group and characterized by a quadratic form. It is as easy as a tensor category can be.
- How much is $\mathcal{U}$ determined by $\mathcal{C}$ and A?

Algebras in Categories

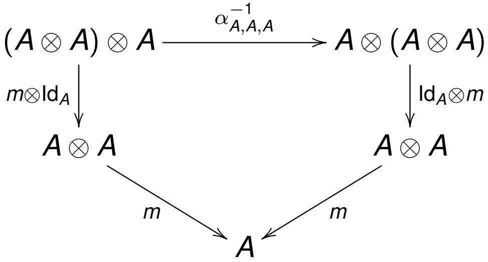
Let $\mathcal{U}$ be a braided tensor category. An **algebra** in $\mathcal{U}$ is an object A in $\mathcal{U}$ together with a multiplication map

$$m : A \otimes A \to A$$

and a unit

$$u : \mathbf{1} \to A$$

such that the multiplication is associative and compatible with left and right multiplication, e.g. the diagram commute:



Commutative Algebras in Categories

The algebra A is called **commutative** if the diagram

$$\begin{array}{ccc} A \boxtimes A & \xrightarrow{c_{A,A}} & A \boxtimes A \\ & {\scriptstyle m} \searrow \quad \swarrow {\scriptstyle m} & \\ & A & \end{array}$$

commutes.

- There is a category $\mathcal{U}_A$ of A-modules in $\mathcal{U}$.
- $\mathcal{U}_A$ is a tensor category but not braided.
- $\mathcal{U}_A$ has a braided tensor subcategory of local modules.

Back to free field realization

- The free field algebra A translates to a commutative algebra in $\mathcal{U}$
- The category $\mathcal{C}$ is precisely the tensor category of local A-modules.
- The aim is to obtain $\mathcal{U}$ from the knowledge of $\mathcal{U}_A$.

Example: $\mathfrak{gl}_{1|1}$

History

- Computes Alexander-Conway polynomials of knots and links (Rozansky-Saleur, 1992).
- A very first example of a logarithmic CFT (Saleur-Schomerus 2005, . . .).
- Used to study disordered systems (LeClair, Ludwig, Saleur, . . .)
- Rigorously only understood now (C-McRae-Yang 2020).

The Lie Algebra

Defining representation, even generators

$$N = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

odd generators

$$\psi^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \psi^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

Relations

$$[N, \psi^\pm] = \pm\psi^\pm, \quad \{\psi^+, \psi^-\} = E.$$

Verma Modules

- Let $V_{n-\frac{1}{2},e}$ for $n, e \in \mathbb{C}$ be the Verma module generated by a highest-weight vector v such that

$$N \cdot v = nv, \quad E \cdot v = ev, \quad \psi^{+} \cdot v = 0.$$

- Since ψ^{-} squares to zero every Verma module has dimension 2; thus n is the average of the two N-eigenvalues of $V_{n,e}$.
- The Verma module $V_{n,e}$ is irreducible if and only if $e \neq 0$. Atypical when $e = 0$, we denote the 1-dimensional irreducible quotient of $V_{n,e}$ by $A_{n+\frac{1}{2}}$.
- For each $n \in \mathbb{C}$, there is a non-split exact sequence

$$0 \to A_{n-\frac{1}{2}} \to V_{n,0} \to A_{n+\frac{1}{2}} \to 0.$$

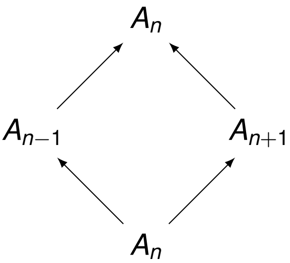
Projective Modules

- For $n \in \mathbb{C}$, the module P_n has basis $v_n, \psi^{\pm} v_n, \psi^+ \psi^- v_n$, where where $E \cdot v_n = 0$ and $N \cdot v_n = n v_n$.
- The module P_n is indecomposable but reducible and satisfies the non-split exact sequence

$$0 \to V_{n+\frac{1}{2},0} \to P_n \to V_{n-\frac{1}{2},0} \to 0.$$

- It has Loewy diagram

P_n:



The affine Lie algebra

- $r, s \in \mathbb{Z}$

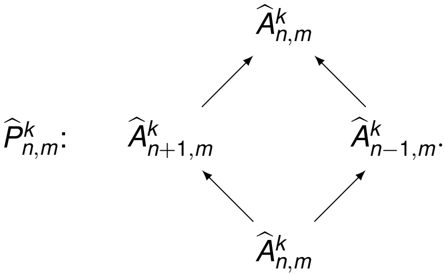
$$[N_r, E_s] = r\mathbf{k}\delta_{r+s,0}, \qquad [N_r, \psi_s^\pm] = \pm\psi_{r+s}^\pm,$$

$$\{\psi_r^+, \psi_s^-\} = E_{r+s} + r\mathbf{k}\delta_{r+s,0},$$

- **k** central
- The zero-mode algebra $< E_0, N_0, \psi_0^\pm >$ is isomorphic to $\mathfrak{gl}_{1|1}$

Modules

- Induced module $\widehat{M}$: M a $\mathfrak{gl}_{1|1}$-module, then for $k \in \mathbb{C}$, let $\mathbf{k}$ act by multiplication of k and X_r by zero for $r \in \mathbb{Z}_{>0}$ and X_s freely for $s \in \mathbb{Z}_{<0}$.
- Modules have a similar structure, except that if $e \in k\mathbb{Z}$, then modules are of atypical type, i.e.



The VOA

- Generating fields $E(z), N(Z), \Psi^{\pm}(z)$.
- Operator products

$$N(z)E(w) \sim \frac{k}{(z-w)^2}, \qquad N(z)\psi^{\pm}(w) \sim \frac{\pm\psi^{\pm}(w)}{(z-w)}$$

$$\psi^{+}(z)\psi^{-}(w) \sim \frac{k}{(z-w)^2} + \frac{E(w)}{(z-w)}$$

The free field realization

- Free bosons $X(z), Y(z)$ and free fermions $b(z), c(z)$
- Operator products

$$X(z)Y(w) \sim \frac{1}{(z-w)^2}, \qquad b(z)c(w) \sim \frac{1}{(z-w)}$$

- The embedding

$$E = kY, \qquad N = X + cb + \frac{1}{2}Y \,,$$

$$\psi^- = b, \qquad \psi^+ = -k\partial c + kcY \,,$$

- It is characterized as the kernel of the zero-mode S_0 (screening charge) of the field $S = be^Y$.

Modules of the free field algebra

- The free fermions are holomorphic, i.e. the VOA is its only simple module
- The free boson simple modules are just Fock modules π_λ for $\lambda \in \mathbb{C}^2$ with fusion rules

$$\pi_\lambda \otimes \pi_\mu \cong \pi_{\lambda+\mu}$$

- The category of modules of the free field algebra is a ribbon category that is equivalent to $\mathcal{C} = \mathrm{Vect}_{\mathbb{C}^2}^{Q} \boxtimes \mathrm{sVect}$ for a certain non-degenerate quadratic form Q.
- This is as easy as a tensor category can be.

The Nichols algebra of screenings

- The screening charge S_0 is naturally associated with the highest-weight vector of a module $x = \pi_\alpha \otimes bc$.
- It satisfies $S_0^2 = 0$.
- It is identified with the algebra $\mathfrak{N} = \mathbb{C}[x]/x^2$, but viewed as an algebra in $\mathcal{C}$.
- $\mathfrak{N}$ is a Hopf algebra (a Nichols algebra) in $\mathcal{C}$ and there is an associated tensor category $\mathrm{Rep}(\mathfrak{N})(\mathcal{C})$.
- Projective modules in this category are of the form

$$0 \to \pi_{\lambda+\alpha} \otimes bc \to P_\lambda \to \pi_\lambda \otimes bc \to 0$$

The category $\mathrm{Rep}(\mathfrak{N})(\mathcal{C})$

- $\mathrm{Rep}(\mathfrak{N})(\mathcal{C})$ is not braided
- The Drinfeld center of a tensor category is always braided, so $\mathcal{Z}(\mathrm{Rep}(\mathfrak{N})(\mathcal{C}))$ is braided.
- $\mathcal{Z}(\mathrm{Rep}(\mathfrak{N})(\mathcal{C}))$ contains $\mathcal{C}^{\mathrm{rev}}$ as a subcategory and its centralizer $\mathcal{Z}_{\mathcal{C}}(\mathrm{Rep}(\mathfrak{N})(\mathcal{C}))$ is braided as well.
- Relative centers $\mathcal{Z}_{\mathcal{C}}(\mathrm{Rep}(\mathfrak{N})(\mathcal{C}))$ can be identified with categories of Yetter-Drinfeld modules ${}^{\mathfrak{N}}_{\mathfrak{N}}\mathcal{YD}(\mathcal{C})$ and the latter allow often for realizing quasi Hopf algebras
- In this case this is $u_q^H(\mathfrak{gl}_{1|1})$ for $q = e^{\pi i/k}$.

The Kazhdan-Lusztig equivalence

Theorem (TC-Lenter-Rupert)

The categories of weight modules of the affine VOA of $\mathfrak{gl}_{1|1}$ at non-zero level k and of $u_q^H(\mathfrak{gl}_{1|1})$ are equivalent as braided tensor categories.

In this case $\mathcal{U}_A \cong \mathrm{Rep}(\mathfrak{N})(\mathcal{C})$.

The Kazhdan-Lusztig equivalence

Remark

In general we prove that a VOA V that embeds in a free field algebra F characterized as kernel of screening charges is equivalent to ${}^{\mathfrak{N}}_{\mathfrak{N}}\mathcal{YD}(\mathcal{C})$ *(and thus the category of its realizing quasi Hopf algebra) provided a series of technical Assumptions can be verified.*

Even if these assumptions can not be verified, the free field realization together with the screenings defines the category ${}^{\mathfrak{N}}_{\mathfrak{N}}\mathcal{YD}(\mathcal{C})$.

What is next?

- Kazhdan-Lusztig correspondence with quantum supergroups will be proven for the boundary VOAs of abelian 3-dimensional $\mathcal{N} = 4$ gauge theories
- The quantum groups will be related to matrix factorization and so will hopefully help us to understand the connection between geometric and VOA interpretations of categories of line operators.
- This is done with Tudor Dimofte and Wenjun Niu.