Singularities of Severi Varieties on K3 Surfaces

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Outline

Severi Varieties

Let X be a smooth projective surface over $\mathbb{C}$. For $L \in \mathrm{Pic}(X)$, let $V_{X,L,g} \subset |L|$ be the Severi variety consisting of integral curves $C \in |L|$ of geometric genus g.

Expected behavior: A general member $C \in V_{X,L,g}$ "should" have exactly

$$\delta = p_a(L) - g = \frac{(K_X + L)L}{2} + 1 - g$$

nodes. The expected dimension of $V_{X,L,g}$ is

$$\dim |L| - \delta = \dim |L| - (p_a(L) - g)$$

Severi Varieties on K3 Surfaces

Let X be a projective K3 surface ($K_X = \mathcal{O}_X$ and $H^1(\mathcal{O}_X) = 0$).
For a big and nef $L \in \operatorname{Pic}(X)$, $V_{X,L,g}$ has the expected dimension

$$\dim |L| - (p_a(L) - g) = g$$

If $V_{X,L,g} \neq \emptyset$, then every irreducible component of $V_{X,L,g}$ has the expected dimension g.

Non-emptiness of $V_{X,L,g}$

(Chen [Che99]) For a very general polarized K3 surface (X, L) over $\mathbb{C}$, every $m \in \mathbb{Z}^+$ and $0 \leq g \leq p_a(mL)$, $V_{X,mL,g} \neq \emptyset$.

(Bogomolov-Tschinkel [BT05], Hassett [Has03], Tayou [Tay18]) $V_{X,L,0} \neq \emptyset$ for infinitely many $L \in \operatorname{Pic}(X)$ on a projective K3 surface X if either X is elliptic or $|\operatorname{Aut}(X)| = \infty$.

(Bogomolov-Hassett-Tschinkel [BHT11]) $V_{X,L,0} \neq \emptyset$ for infinitely many $L \in \operatorname{Pic}(X)$ on a projective K3 surface X if X has genus 2 and $\operatorname{rank} \operatorname{Pic}(X) = 1$.

Non-emptiness of $V_{X,L,g}$

(Li-Liedtke [LL11]) $V_{X,L,0} \neq \emptyset$ for infinitely many $L \in \mathrm{Pic}(X)$ on a projective K3 surface X if $\operatorname{rank} \mathrm{Pic}(X)$ odd.

(Chen-Gounelas-Liedtke [CGL19a], Chen-Gounelas [CG20]) For every $g \geq 0$, $V_{X,L,g} \neq \emptyset$ for infinitely many $L \in \mathrm{Pic}(X)$ on every projective K3 surface X over $\mathbb{C}$.

Conjecture. $V_{X,L,g} \neq \emptyset$ for every projective K3 surface X over $\mathbb{C}$, every very ample $L \in \mathrm{Pic}(X)$ and $0 \leq g \leq p_a(L)$.

Irreducibility of Severi Varieties on K3 surfaces

Conjecture. For a general polarized K3 surface (X, L) and all $1 \leq g \leq p_a(L)$, $V_{X,L,g}$ is irreducible.

True if $\delta = p_a(L) - g$ is "small".

Irreducibility of Severi Varieties of Plane Curves

(Harris [Har86]) The Severi variety $V_{d,g}$ of plane curves of genus g is irreducible.

Harris' proof consists of two parts

Easy: $V_{d,0}$ is irreducible and the monodromy group of

$$W_{d,0} = \{(C, p) : C \in V_{d,0} \text{ and } p \in C_{\text{sing}}\}$$

over $V_{d,0}$ is the full symmetric group.

Hard: $V_{d,0} \subset \overline{V}$ for every irreducible component V of $V_{d,g}$.

Irreducibility of Severi Varieties on K3 surfaces

If we mimic Harris' proof, we need to do

Hard: $V_{X,L,1}$ is irreducible and the monodromy group of

$$W_{X,L,1} = \{(C,p) : C \in V_{X,L,1} \text{ and } p \in C_{\text{sing}}\}$$

over $V_{X,L,1}$ is the full symmetric group.

Easy: $V_{X,L,1} \subset \overline{V}$ for every irreducible componet V of $V_{X,L,g}$ and all $g \geq 1$ [Che19].

Irreducibility of Severi Varieties on K3 surfaces

(A. Bruno and M. Lelli-Chiesa [BLC21]) For a general polarized K3 surface (X, L), $V_{X,L,g}$ is connected for all $1 \leq g \leq p_a(L)$ and is irreducible for all $p_a(L) \geq 5$ and $g \geq 4$.

New techniques:

1. Induction from large g to small g.
2. Derive "irreducibility" from "connectedness".

Connectedness and Irreducibility

Let V be a variety. Then

$$\left.\begin{array}{l} V \text{ is connected} \\ V \text{ is Cohen-Macaulay} \\ V \text{ is smooth in codimension 1} \end{array}\right\} \xrightarrow{\text{Hartshorne's Connectedness}} V \text{ is irreducible}$$

Let $V = \cup V_i$ for irreducible components V_i of V. For $V_i \neq V_j$, since V is smooth in codimension 1, $\operatorname{codim}_V(V_i \cap V_j) \geq 2$. By Harshorne's Connectedness,

$$V \backslash \bigcup_{i \neq j} (V_i \cap V_j)$$

is connected.

Moduli Stack of Stable Maps to K3 Surfaces

For a polarized K3 surface (X, L), let

$$\overline{\mathcal{M}}_{X,L,g} = \{f : C \to X \text{ stable map of genus } g,\ f_* C \in |L|\}$$

be the moduli stack of stable maps of genus g to X and let

$$\mathcal{M}_{X,L,g} = \rho^{-1}(V_{X,L,g})$$

under the map $\rho : \overline{\mathcal{M}}_{X,L,g} \to |L|$ sending $[f : C \to X]$ to $f_* C \in |L|$.

The morphism $\rho : \mathcal{M}_{X,L,g} \to V_{X,L,g}$ is one-to-one, onto and birational. But it is NOT an isomorphism.

$V_{X,L,g}$ is connected/irreducible if and only if $\mathcal{M}_{X,L,g}$ is.

Tangent Space of $\mathcal{M}_{X,L,g}$

The Zariski tangent space to $\mathcal{M}_{X,L,g}$ at $[f : C \to X]$ is

$$\mathrm{Ext}([f^*\Omega_X \to \Omega_C], \mathcal{O}_C) = H^0(N_f)$$

with obstruction

$$\mathrm{Ext}^2([f^*\Omega_X \to \Omega_C], \mathcal{O}_C) = H^1(N_f)$$

for $N_f = \mathrm{coker}(T_C \to f^* T_X)$.

Dimension of $\mathcal{M}_{X,L,g}$

If $f^*\Omega_X \to \Omega_C$ is surjective, $N_f \cong K_C$ and

$$\dim_{[f]} \mathcal{M}_{X,L,g} \leq h^0(N_f) = g$$

By Arbarello-Cornalba [AC81, Lemma 1.4], for $[f] \in \mathcal{M}_{X,L,g}$ general,

$$\dim_{[f]} \mathcal{M}_{X,L,g} \leq h^0(N_f/(N_f)_{\text{tors}}) \leq g.$$

$\mathcal{M}_{X,L,g}$ is a local complete intersection

(Twisted Family) Let Y/Δ be a smooth family of complex K3 surfaces over the unit disk $\Delta = \{|t| < 1\}$ such that $Y_0 = X$ and Y_t is a complex K3 surface with $\mathrm{Pic}(Y_t) = 0$ for $t \neq 0$.

We fix a closed embedding $\rho : C \hookrightarrow P = \mathbb{P}^n$ such that $H^1(\rho^* T_P) = 0$. Let W be the connected component of the Hilbert scheme $\mathrm{Hilb}(P \times Y/\Delta)$ containing $[\Gamma]$ for $\Gamma = (\rho \times f)(C) \subset P \times Y_0$. Then W is smooth over $\mathcal{M}_{X,L,g}$ of dimension

$$\dim W \leq \dim \mathcal{M}_{X,L,g} + h^0(\rho^* T_P) = g + h^0(\rho^* T_P)$$

Also

$$\dim W \geq \chi(N_{\rho \times f}) + \dim \Delta = g + h^0(\rho^* T_P)$$

So W is a local complete intersection of dimension $g + h^0(\rho^* T_P)$.

Smooth locus of $\mathcal{M}_{X,L,g}$

So $\mathcal{M}_{X,L,g}$ is a local complete intersection of dimension g.

If $f^*\Omega_X \to \Omega_C$ is surjective (f is unramified), $\mathcal{M}_{X,L,g}$ is smooth at $[f]$.

If f is unramified outside of a double point,

$$N_f \cong K_C(-p) \oplus \mathcal{O}_p$$

and $h^0(N_f) = g$. Hence $\mathcal{M}_{X,L,g}$ is smooth at $[f]$.

Singularities of $\mathcal{M}_{X,L,g}$ and $V_{X,L,g}$

(Work in progress) For a general polarized K3 surface (X, L) over $\mathbb{C}$ and every $1 \leq g \leq p_a(L)$, fixing $g - 1$ general points $p_1, p_2, ..., p_{g-1}$ on X, every curve $C \in |L|$ of (geometric) genus g passing through $p_1, p_2, ..., p_{g-1}$ has at least $p_a(L) - g - 1$ nodes, where $p_a(L) = (L^2 + 2)/2$ is the arithmetic genus of L.

(Chen [Che02]) All rational curves in $|L|$ are nodal for a general polarized K3 surface (X, L).

Singularities of $\mathcal{M}_{X,L,g}$ and $V_{X,L,g}$

Corollary. $\mathcal{M}_{X,L,g}$ is smooth in codimension one.

Corollary. $\mathcal{M}_{X,L,g}$ is a normal local complete intersection.

Corollary. The codimension one singular locus of $V_{X,L,g}$ is cuspidal.

Corollary. $\mathcal{M}_{X,L,g}$ and $V_{X,L,g}$ are connected if and only if they are irreducible.

Degeneration to Bryan-Leung K3

Let $\pi : X \to \Delta$ be a one-parameter family of K3 surfaces of genus n over the unit disk $\Delta = \{|t| < 1\}$ such that X_0 is a Bryan-Leung K3 surface with Picard group generated by C and F with intersection matrix

$$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

The main advantage to work with these K3 surfaces is that the linear system $|C + nF|$ breaks up into

$$H^0(X_0, C + nF) = H^0(X_0, C) \otimes \mathrm{Sym}^n H^0(X_0, F)$$

That is, every curve $R \in |C + nF|$ is

$$R = C + F_1 + F_2 + \ldots + F_n$$

for some $F_i \in |F|$.

Degeneration to Bryan-Leung K3

We fix $g-1$ general sections of X/Δ and let P be the union of these sections.
Suppose that there is a family $f : \mathscr{C}/\Delta \to X/\Delta$ of stable maps of genus g, after a possible base change, such that $\mathscr{C}_t$ is smooth, $f_*\mathscr{C}_t \in |L_t|$ and $f(\mathscr{C}_t)$ contains P_t for $t \in \Delta$.
It suffices to prove that $f(\mathscr{C}_t)$ has at worst a cusp for $t \neq 0$. Suppose that

$$\begin{aligned} f_*\mathscr{C}_0 = C + m_1F_1 + m_2F_2 + \dots + m_{g-1}F_{g-1} + m_gF_g \\ + n_1G_1 + n_2G_2 + \dots + n_{24}G_{24} \end{aligned}$$

where $G_1, G_2, ..., G_{24}$ are 24 nodal rational curves in $|F|$ and $F_1, F_2, ..., F_g$ are smooth members in $|F|$.

Local Deformation Theory

(Ran, Caporaso-Harris) Let X be the 3-fold in Δ^4_{xyzt} given by $xy = t^\alpha$ for some positive integer α and let $Y \subset X$ be a flat family of curves in X such that $Y_0 = C_1 \cup C_2$ is a union of two smooth curves

$$C_1 \subset R_1 = \{x = t = 0\} \text{ and } C_2 \subset R_2 = \{y = t = 0\}$$

with each C_i tangent to the curve

$$D = \{x = y = t = 0\}$$

in R_i with multiplicity $m \in \mathbb{Z}^+$ at the origin. Suppose that the total δ-invariant of Y_t is $m - 1$ for $t \neq 0$. Then

- Y_t is nodal for $t \neq 0$, and
- α is divisible by m.

Local Deformation Theory

Let X be the 3-fold in Δ^4_{xyzt} given by $xy = t^\alpha z$ for some positive integer α and let $Y \subset X$ be a flat family of curves in X such that $Y_0 = C_1 \cup C_2$ is a union of two smooth curves

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- Y_t has at worst a cusp for $t \neq 0$, and
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Enrico Arbarello and Maurizio Cornalba.
Footnotes to a paper of Beniamino Segre: "On the modules of polygonal curves and on a complement to the Riemann existence theorem" (Italian) [Math. Ann. **100** (1928), 537–551; Jbuch **54**, 685].
Math. Ann., 256(3):341–362, 1981.
The number of g_d^1's on a general d-gonal curve, and the unirationality of the Hurwitz spaces of 4-gonal and 5-gonal curves.

Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths.
Geometry of algebraic curves. Volume II, volume 268 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*.
Springer, Heidelberg, 2011.
With a contribution by Joseph Daniel Harris.

Fedor Bogomolov, Brendan Hassett, and Yuri Tschinkel.
Constructing rational curves on K3 surfaces.
Duke Mathematical Journal, 157(3):535–550, April 2011.

Jim Bryan and Naichung Conan Leung.
The enumerative geometry of K3 surfaces and modular forms.
Journal of the American Mathematical Society, 13(2):371–410, 2000.

Andrea Bruno and Margherita Lelli-Chiesa.
Irreducibility of Severi varieties on K3 surfaces.
2021.

Fedor Bogomolov and Yuri Tschinkel.
Rational curves and points on K3 surfaces.
Amer. J. Math., 127(4):825–835, 2005.

Xi Chen and Frank Gounelas.

Curves of maximal moduli on K3 surfaces.
arXiv.org, page arXiv:2007.01735, Jul 2020.

Xi Chen, Frank Gounelas, and Christian Liedtke.
Curves on K3 surfaces.
arXiv.org, page arXiv:1907.01207, Jul 2019.

Xi Chen, Frank Gounelas, and Christian Liedtke.
Rational curves on lattice-polarised K3 surfaces.
arXiv.org, page arXiv:1907.01207, Jul 2019.

Xi Chen.
Rational curves on K3 surfaces.
J. Algebraic Geom., 8(2):245–278, 1999.

Xi Chen.
A simple proof that rational curves on K3 are nodal.
Math. Ann., 324(1):71–104, Jul 2002.
Also preprint arXiv:math/0011190.

Xi Chen.
Nodal Curves on K3 Surfaces.
New York J. Math., 25:168–173, 2019.
Also preprint arXiv:1611.07423.

Xi Chen and James D. Lewis.
Density of rational curves on K3 surfaces.
Math. Ann., 356(1):331–354, 2013.

Joe Harris.
On the severi problem.
Inventiones mathematicae, 84(3):445–461, Oct 1986.

Brendan Hassett.
Potential density of rational points on algebraic varieties.
In *Higher dimensional varieties and rational points (Budapest, 2001)*, volume 12 of *Bolyai Soc. Math. Stud.*, pages 223–282. Springer, Berlin, 2003.

Joe Harris and Ian Morrison.
Moduli of curves, volume 187 of *Graduate Texts in Mathematics*.
Springer-Verlag, New York, 1998.

Daniel Huybrechts.
Lectures on K3 surfaces, volume 158 of *Cambridge Studies in Advanced Mathematics*.
Cambridge University Press, Cambridge, 2016.

Vik. S. Kulikov.
Degenerations of K3 surfaces and Enriques surfaces.
Izv. Akad. Nauk SSSR Ser. Mat., 41(5):1008–1042, 1199, 1977.

Jun Li and Christian Liedtke.
Rational curves on K3 surfaces.
Inventiones mathematicae, 188(3):713–727, October 2011.

Shigefumi Mori and Shigeru Mukai.

The uniruledness of the moduli space of curves of genus 11.
In *Algebraic geometry (Tokyo/Kyoto, 1982)*, volume 1016 of *Lecture Notes in Math.*, pages 334–353. Springer, Berlin, 1983.

Ben Moonen.
The Deligne-Mostow List and Special Families of Surfaces.
International Mathematics Research Notices, 2018(18):5823–5855, 03 2017.

Viacheslav V. Nikulin.
Elliptic fibrations on K3 surfaces.
Proc. Edinb. Math. Soc. (2), 57(1):253–267, 2014.

Salim Tayou.
Rational curves on elliptic K3 surfaces.
arXiv.org, page arXiv:1805.07975, May 2018.