Quivers, Flow Trees, and Log Curves

Hülya Argüz

University of Georgia

Hodge Theory, Mirror Symmetry and Physics of Calabi-Yau Moduli
University of Heidelberg, 15 June 2023

Plan of the talk

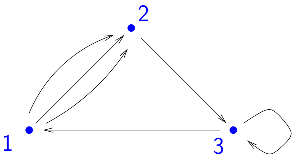
- Algebra: Quiver Donaldson–Thomas (DT) Invariants
  - The attractor flow tree formula (calculating quiver DT invariants via tropical geometry)
- Geometry: Counts of log curves in toric varieties
  - From quivers to toric varieties
  - Log Gromov–Witten (GW) invariants of toric varieties
  - Calculating log GW invariants tropically

- Quiver DT invariants $\longleftrightarrow$ log GW invariants of toric varieties

Definition

A *quiver* is a finite oriented graph $Q = (Q_0, Q_1, s, t)$.

- Q_0: set of vertices.
- Q_1: set of arrows.
- $s : Q_1 \to Q_0$ maps an arrow to its *source*.
- $t : Q_1 \to Q_0$ maps an arrow to its *target*.



$$Q_0 = \{1, 2, 3\}$$

Representations of Quivers

Definition

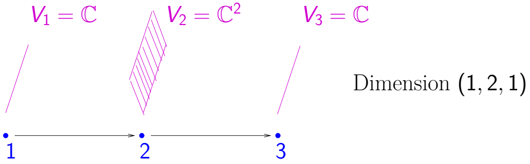
A *representation of a quiver* is an assignement of

- a vector space V_v, for each vertex $v \in Q_0$, and
- a linear transformation $f_{ij} \in \mathrm{Hom}_{\mathbb{C}}(V_{s(e)}, V_{t(e)})$ for each edge $e \in Q_1$.

- *Dimension* of a quiver representation is a vector

$$\gamma = (\gamma_i)_{i \in Q_0} \in N^+,$$

where $N := \mathbb{Z}^{Q_0}$ and $N^+ = \mathbb{N}^{Q_0} \setminus \{0\}$, encoding dimensions of the vector spaces assigned to vertices.



Representations of Quivers

- There is a natural notion of morphisms/isomorphisms between two quiver representations (f_{ij}) and (g_{ij}):
  - automorphisms $h_i \colon \mathbb{C}^{\gamma_i} \to \mathbb{C}^{\gamma_i}$ such that $g_{ij} = f_{ij} \circ h_i$.

Definition (King's notion of stability)

- V: quiver representation of dimension $\gamma \in N^+$.
- $M := \mathrm{Hom}(N, \mathbb{Z})$ and $M_{\mathbb{R}} = \mathrm{Hom}(N, \mathbb{R}) = M \otimes \mathbb{R}$
- $\theta \in \gamma^{\perp} := \{\theta \in M_{\mathbb{R}}, \theta(\gamma) = 0\} \subset M_{\mathbb{R}}$: **stability parameter**.
  - V: *θ-stable* if $\forall \{0\} \subsetneq V' \subsetneq V$ we have $\theta(\dim(V')) < 0$.
  - V: *θ-semi-stable* if $\forall V' \subsetneq V$ we have $\theta(\dim(V')) \leq 0$.

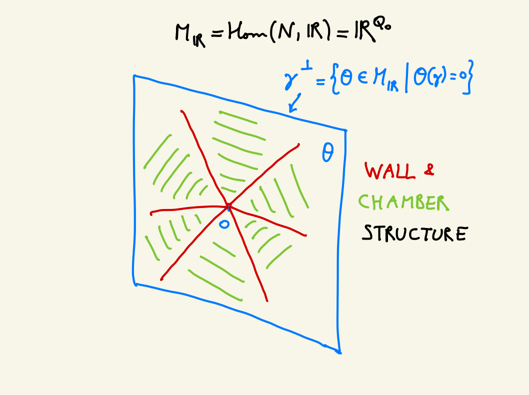
- $\mathcal{M}_{\gamma}^{\theta}$: Moduli space of θ semi-stable quiver representations of Q dimension γ.

Quiver DT invariants

- "In nice cases" (when $\mathcal{M}_\gamma^\theta$: smooth) we define quiver DT invariants as the topological Euler characteristics:

$$DT_\gamma^\theta := e(\mathcal{M}_\gamma^\theta) = \sum_k (-1)^k \dim H^k(\mathcal{M}_\gamma^\theta, \mathbb{C}) \,.$$

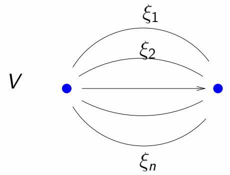
- Piecewise constant dependence on $\theta \in \gamma^\perp$: wall-crossing, universal wall-crossing formula (Kontsevich-Soibelman).



Example

Example

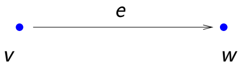
- Q: n-Kronecker quiver
- V: representation with $\gamma := \dim(V) = (1,1) \in N$
- $\theta = (\theta_1, -\theta_1) \in \gamma^{\perp} \subset M_{\mathbb{R}}$.



- $\theta_1 > 0$ and $(\xi_1, \ldots, \xi_n) \neq 0 \implies V$ is θ semi-stable, $\mathcal{M}_{\gamma}^{\theta} \cong \mathbb{CP}^{n-1}$
- $\theta_1 < 0 \implies \mathcal{M}_{\gamma}^{\theta} = \emptyset$.

Quivers with potentials

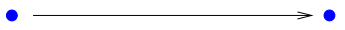
- Path algebra $\mathbb{C}Q$: $\mathbb{C}$-linear combinations of paths in Q with concatenation product.



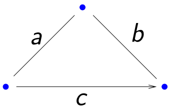
$$\mathbb{C}Q = \mathbb{C}v \oplus \mathbb{C}e \oplus \mathbb{C}w$$
$$v^2 = v, \;\; w^2 = w$$
$$ev = we = e$$

- Potential $W \in \mathbb{C}Q$: Formal linear combination of oriented cycles.

Acyclic Quiver



$W = 0$



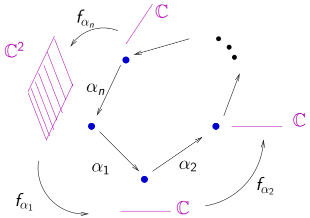
$W = 2abc + 5(abc)^2$

Not allowed!

- We assume quivers do not have oriented two-cycles.

The trace function

- For $(Q, W = \sum \lambda_c c)$ define the **trace function**



$$\mathrm{Tr}(c)^\theta_\gamma : \mathcal{M}^\theta_\gamma \to \mathbb{C}$$

$$V = ((V_i)_{i \in Q_0}, (f_\alpha)_{\alpha \in Q_1}) \longmapsto \mathrm{Tr}(f_{\alpha_n} \circ \ldots \circ f_{\alpha_1})$$

$$\mathrm{Tr}(W)^\theta_\gamma = \sum_c \lambda_c \mathrm{Tr}(c)^\theta_\gamma$$

- C^θ_γ: Critical locus of $\mathrm{Tr}(W)^\theta_\gamma \subset \mathcal{M}^\theta_\gamma$.
- "In nice cases" ($\mathcal{M}^\theta_\gamma$ smooth and $\mathrm{Tr}(W)^\theta_\gamma$ Morse-Bott)

$$\Omega^\theta_\gamma := e(C^\theta_\gamma) = \sum_k (-1)^k \dim H^k(C^\theta_\gamma, \mathbb{C}) .$$

The (general) definition of DT invariants

Definition

For (Q, W): quiver with potential, $\gamma \in N^+$, and $\theta \in \gamma^\perp \subset M_{\mathbb{R}}$, the **Donaldson–Thomas (DT) invariant** $\Omega_\gamma^\theta \in \mathbb{Z}$ for $((Q, W), \gamma, \theta)$ is defined by

$$\Omega_\gamma^\theta = e(C_\gamma^\theta, \phi_{\mathrm{Tr}(W)_\gamma^\theta} \mathcal{IC}_{M_\gamma^\theta})$$

- $\mathcal{IC}_{M_\gamma^\theta}$: intersection cohomology sheaf on M_γ^θ
  - $\mathcal{IC}_{M_\gamma^\theta}$ is a perverse sheaf (M_γ^θ smooth $\implies \mathcal{IC}_{M_\gamma^\theta}$ is the constant sheaf with stalk $\mathbb{Q}$)
- $\phi_{\mathrm{Tr}(W)_\gamma^\theta}$: *vanishing cycle functor* for the function $\mathrm{Tr}(W)_\gamma^\theta$
  - $\phi_{\mathrm{Tr}(W)_\gamma^\theta} \mathcal{IC}_{M_\gamma^\theta}$: sheaf on the critical locus $C_\gamma^\theta \subset M_\gamma^\theta$
- We will work with **rational DT invariants** given by

$$\overline{\Omega}_\gamma^\theta := \sum_{\substack{\gamma = k\gamma' \\ k \in \mathbb{Z}_{\geq 1}, \gamma' \in N^+}} \frac{(-1)^{k-1}}{k^2} \Omega_{\gamma'}^\theta$$

- See Kontsevich–Soibelman, Joyce–Song, Reineke, Davison–Meinhardt

Ex: Ω_γ^θ can generally be very complicated

- The 3-Kronecker quiver appears in $\mathcal{N} = 2$, $4d$ $SU(3)$ super Yang-Mills theory[1]

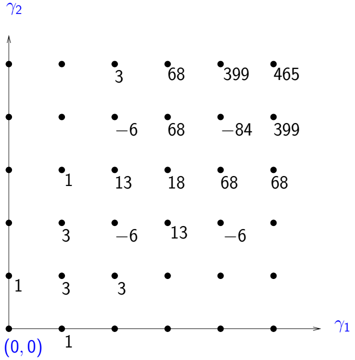
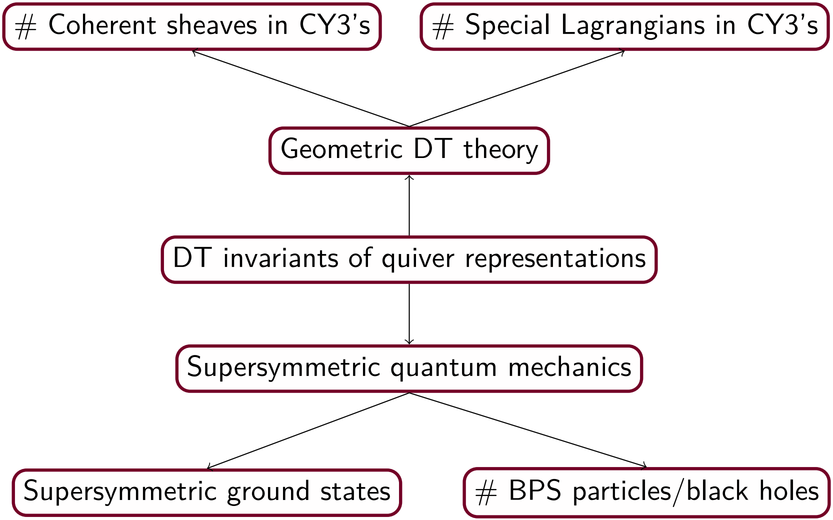


Figure: Values of Ω_γ^θ for the 3-Kronecker quiver

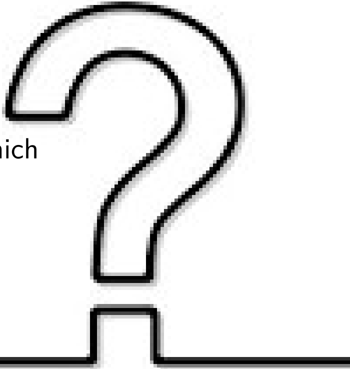
[1] Galakhov–Longhi–Mainiero–Moore–Neitzke, "Wild wall crossing and BPS giants." Journal of High Energy Physics 2013.

Why are refined DT invariants of quivers interesting?



DT invariants from a “simple” set of invariants

Is there a primitive set of DT invariants from which we could determine all DT invariants?



- Yes! We calculate quiver DT invariants using wall structures and flow trees, from simpler (attractor) DT invariants.

H. Argüz, P. Bousseau: *The flow tree formula for Donaldson–Thomas invariants of quivers with potentials*, Compositio Mathematica 158 (12), 2206-2249, 2022

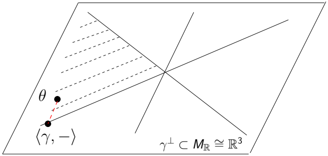
A simple set of DT invariants: attractor DT invariants

- Let $\{s_1, \ldots, s_{|Q_0|}\}$ be a basis for N. Define a skew symmetric form $\langle -, - \rangle$ on N by
$$\langle s_i, s_j \rangle := a_{ij} - a_{ji}.$$
where a_{ij} is the number of arrows from i to j.
- Fix $\gamma \in N$. The chamber containing $\langle \gamma, - \rangle \in \gamma^{\perp} \in M_{\mathbb{R}}$ is an attractor chamber for γ (generally not γ-generic).

Definition (Alexandrov–Pioline, Mozgovoy–Pioline, Kontsevich–Soibelman)

Let $\theta \in \gamma^{\perp} \subset M_{\mathbb{R}}$ be a small perturbation of $\langle \gamma, - \rangle$ which is γ-generic. Define the **attractor DT invariants** by $\Omega^{\star}_{\gamma} := \Omega^{\theta}_{\gamma}$.

- $\Omega^{\star}_{\gamma}$ do not depend on the stability parameter θ, and are generally much simpler to compute.



The attractor DT invariants

Theorem (Bridgeland[a])

[a] Generalizations for some non-acyclic quivers: Lang Mou, arXiv:1910.13714

If Q is acyclic then

$$\Omega^\star_\gamma = \begin{cases} 1 & \text{if } \gamma = (0,\dots,0,1,0,\dots,0) \\ 0 & \text{otherwise} \end{cases}$$

Conjecture (Beaujard–Manschot–Pioline, Mozgovoy–Pioline[a])

[a] Proven for the local P2 by Bousseau–Descombes–Le Floch–Pioline, arxiv:2210.10712

- *K_S: local del Pezzo (canonical bundle over a del Pezzo surface S)*
- *Q: quiver, with the additional data of a potential functions W s.t. $D^b Rep(Q,W) \cong D^b Coh(K_S)$, then $\Omega^\star_\gamma = 0$, unless either $\gamma = (0,\dots,0,1,0,\dots,0) \implies \Omega^\star_\gamma = 1$ or γ is a multiple of a class of a point, in which case $\Omega^\star_\gamma$ equals the Euler characteristic of S.*

Attractor flows (Kontsevich-Soibelman arXiv:1303.3253)

- (Q, W): quiver with potential
- $\gamma \in N^+$, and $\theta \in \gamma^\perp$, γ-generic.
- Iterative application of the Kontsevich-Soibelman wall-crossing formula:

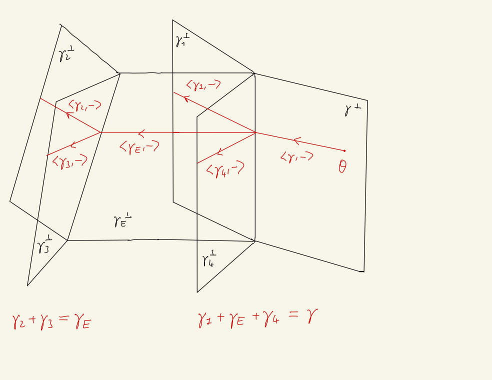
$$\overline{\Omega}^{\theta}_{\gamma} = \sum_{\gamma=\gamma_1+\cdots+\gamma_r} \frac{1}{|\mathrm{Aut}((\gamma_i)_i)|} F^{\theta}_{r}(\gamma_1, \dots, \gamma_r) \prod_{i=1}^{r} \overline{\Omega}^{\star}_{\gamma_i} .$$

where

- $|\mathrm{Aut}((\gamma_i)_i)|$ is the order of the group of permutation symmetries of the decomposition $\gamma = \gamma_1 + \cdots + \gamma_r$, and
- The coefficients $F^{\theta}_{r}(\gamma_1, \dots, \gamma_r)$ are sums of contributions from attractor trees with leaves decorated by $\gamma_1, \dots, \gamma_r$ and with root at θ:

$$F^{\theta}_{r}(\gamma_1, \dots, \gamma_r) = \sum_{T \in \mathcal{T}^{\theta}_{\gamma_1,\dots,\gamma_r}} F^{\theta}_{r,T}(\gamma_1, \dots, \gamma_r)$$

M. Kontsevich, Y. Soibelman: "Wall-crossing structures in Donaldson–Thomas invariants, integrable systems and mirror symmetry". In Homological mirror symmetry and tropical geometry (pp. 197-308), Springer, 2014



- Attractor trees are in particular **tropical trees** – they satisfy the "tropical balancing condition" (weighted directions around edges add up to zero)

The flow tree formula

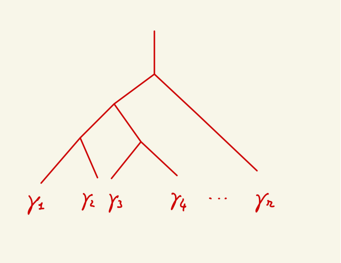
Theorem (Flow tree formula (A-Bousseau))

- (Q, W): *quiver with potential*
- $\gamma \in N^+$, *and* $\theta \in \gamma^\perp$, γ*-generic.*

$$\overline{\Omega}_\gamma^\theta = \sum_{\gamma=\gamma_1+\cdots+\gamma_r} \frac{1}{|\mathrm{Aut}((\gamma_i)_i)|} F_r^\theta(\gamma_1, \ldots, \gamma_r) \prod_{i=1}^r \overline{\Omega}_{\gamma_i}^\star .$$

where

- $F_r^\theta(\gamma_1, \ldots, \gamma_r) \in \mathbb{Q}$ *are described concretely in terms of* ***binary*** *trees.*



- Binary attractor trees are perturbations of the (generally non-binary) atttactor trees with roots at θ.
- Conjectured by Alexandrov-Pioline.
- Proof uses wall structures.
- A variant of this formula is proven by Mozgovoy using operads.

The coefficients $F_r^\theta(\gamma_1, \dots, \gamma_r)$

- For (Q, W), let $\gamma = \gamma_1 + \cdots + \gamma_r \in N^+$. (repetitions allowed!)
- Simplifying assumption for now: $\{\gamma_1, \dots, \gamma_r\}$ is a basis for N.

$$F_r^\theta(\gamma_1, \dots, \gamma_r) := \sum_{T_r} \prod_{v \in V_{T_r}^\circ} -\epsilon_{T_r,v}^{\widetilde{\theta}} \langle \gamma_{v'}, \gamma_{v''} \rangle \,.$$

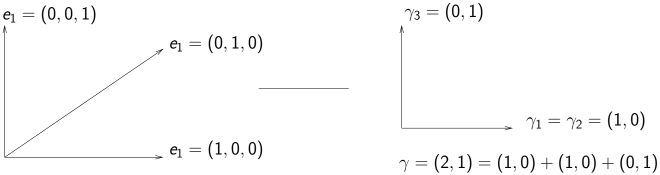
- T_r: rooted binary trees with r leaves (decorated by $\{\gamma_1, \dots, \gamma_r\}$),
- $V_{T_r}^\circ$: set of interior vertices of of T_r,
- $\gamma_v \in \mathcal{N}$ is the sum of γ_i's attached to leaves descendant from v for any $v \in V_{T_r}^\circ$,
- $\widetilde{\theta}$ is a small generic perturbation of θ in $M_{\mathbb{R}}$
- $\epsilon_{T_r,v}^{\widetilde{\theta}} \in \{-1, 0, 1\}$ is a sign defined via "flows" (these signs control the realizability of T_r as a binary attractor tree with root at $\widetilde{\theta}$.).

The general case

- Generally, for $\gamma = \gamma_1 + \ldots + \gamma_r$, if $\{\gamma_1, \ldots, \gamma_r\}$ is not a basis, we introduce a bigger lattice

$$\mathcal{N} := \bigoplus_{i=1}^{r} \mathbb{Z} e_i$$

  and consider the map
- $p \colon \mathcal{N} \to N$ defined by $e_i \mapsto \gamma_i$
- Define a skew-symmetric form η on $\mathcal{N}$ by $\eta(e_i, e_j) := \langle \gamma_i, \gamma_j \rangle$.

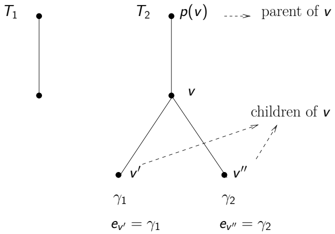


- In this bigger space we can work with perturbations of attractor trees into binary trees.

Example: for Q the n-Kronecker quiver

- Q the n-Kronecker quiver
- Let $\theta = (\theta_1, -\theta_1)$ and $\gamma = (1,1)$, so that $\gamma_1 = (1,0)$, $\gamma_2 = (0,1)$. In this case, can actually take $\widetilde{\theta} = \theta$.

$$F_1^\theta(\gamma_1, \gamma_2) = 1$$
$$F_2^\theta(\gamma_1, \gamma_2) = -\epsilon^\theta_{T,v} n$$



- We have $\theta_1 < 0 \implies \epsilon^\theta_{T,v} = 0$ and $\theta_1 > 0 \implies \epsilon^\theta_{T,v} = -1$

$$\begin{aligned}\overline{\Omega}^\theta_\gamma &= F_1^\theta(\gamma)\overline{\Omega}^*_\gamma + F_2^\theta(\gamma_1,\gamma_2)\overline{\Omega}^*_{\gamma_1}\overline{\Omega}^*_{\gamma_2} \\ &= 1 \cdot 0 - (-1)n \\ &= n\end{aligned}$$

GOAL: Quivers to Curves

A correspondence between

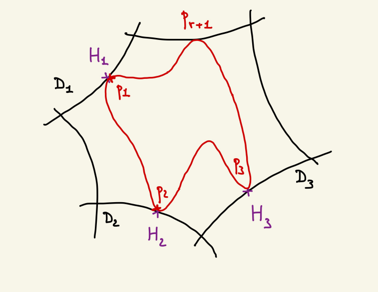
$$F_r^{\theta}(\gamma_1, \ldots, \gamma_r)$$

and counts of rational curves in a toric variety X_{Σ}.

The toric variety X_Σ and enumerative geometry

Fix a quiver Q, a dimension vector $\gamma \in N$. For any decomposition $\gamma = \gamma_1 + \ldots + \gamma_r$, we set

- Σ : a fan in $M_{\mathbb{R}}$ of a smooth projective toric variety X_Σ containing the rays $\mathbb{R}_{\geq 0}\langle \gamma_i, - \rangle$ for all $1 \leq i \leq r$.
- $H_i \subset D_i$ hypersurfaces defined by $\{z^{\gamma_i^{prim}} = constant\}$

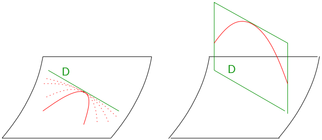


- Count genus 0 stable maps $(C, \{p_1, \ldots, p_{r+1}\}) \to X_\Sigma$ satisfying
  - $p_i \mapsto H_i$ for all $1 \leq i \leq r$
  - The contact order of the image of p_i with D_i is the divisibility of $\langle \gamma_i, - \rangle$

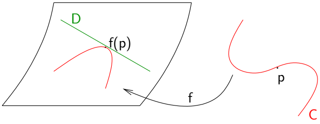
Log Gromov–Witten theory

- Jun Li: The case $D \subset X$ is smooth. Expand the target;

$$
\begin{aligned}
X &\mapsto X[1] = X \amalg_D \mathbb{P}(\mathcal{N}_{D|X} \oplus \mathcal{O}_D) \\
&\mapsto X[2] = X[1] \amalg_D \mathbb{P}(\mathcal{N}_{D|X} \oplus \mathcal{O}_D) \\
&\mapsto X[3] = \dots
\end{aligned}
$$



- Gross–Siebert/Abramovich–Chen: The case $D \subset X$ is log smooth. Record contact orders using "log structures"



Log geometry

Definition

A **log structure** on X is a sheaf of monoids $\mathcal{M}_X$ together with a map $\alpha : \mathcal{M} \longrightarrow (\mathcal{O}_X, \cdot)$ inducing an isomorphism $\alpha^{-1}(\mathcal{O}_X^\times) \simeq \mathcal{O}_X^\times$. A **log scheme** $(X, \mathcal{M}_X)$ is a scheme with a log structure.

Definition

The **ghost sheaf** of a log scheme $(X, \mathcal{M}_X)$ is the sheaf of monoids

$$\overline{\mathcal{M}}_X := \mathcal{M}_X / \alpha(\mathcal{O}_X^\times).$$

Definition

The **tropicalization** $\Sigma(X)$ of a log scheme $(X, \mathcal{M}_X)$ is the cone complex

$$\coprod_\eta (\overline{\mathcal{M}}_{X,\eta})^\vee_{\mathbb{R}} := \mathrm{Hom}(\overline{\mathcal{M}}_{X,\eta}, \mathbb{R}_{\geq 0}) / \sim$$

indexed by the generic points η of the log strata of X.

Log geometry

Example (The divisorial log structure)

Let $D \subset X$ be a divisor, and $j : X \setminus D \to X$. Define $\mathcal{M}_{(X,D)} := j_*(\mathcal{O}^{\times}_{X\setminus D}) \cap \mathcal{O}_X$, and $\alpha_X : \mathcal{M}_{(X,D)} \hookrightarrow \mathcal{O}_X$ to be the inclusion.

- $\mathcal{M}_{\mathbb{A}^1_t,0} = \{h \cdot t^n \mid h \in \mathcal{O}^{\star}_{\mathbb{A}^1}\}$.
- $\overline{\mathcal{M}}_{\mathbb{A}^1_t,0,0} \cong \mathbb{N}$, via the isomorphism $t^n \mapsto n$.

Example (The standard log point)

Let $X := \operatorname{Spec} \mathbb{C}$, $\mathcal{M}_X := \mathbb{C}^{\times} \oplus \mathbb{N}$, and define $\alpha_X : \mathcal{M}_X \to \mathbb{C}$ as follows:

$$\alpha_X(x,n) := \begin{cases} x & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

Stable log maps

Let $(S, \mathcal{M}_S)$ be a log point and let $(X, \mathcal{M}_X)$ be a log scheme over $(S, \mathcal{M}_S)$ (in applications, $(S, \mathcal{M}_S)$ will be either the trivial log point or the standard log point).

Definition

A **stable log map** with target X/S is a commutative diagram

$$\begin{array}{ccc} (C, \mathcal{M}_C) & \xrightarrow{f} & (X, \mathcal{M}_X) \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ (W, \mathcal{M}_W) & \longrightarrow & (S, \mathcal{M}_S) \end{array}$$

where $(W, \mathcal{M}_W)$ is a log point, and $\pi\colon (C, \mathcal{M}_C) \to (W, \mathcal{M}_W)$ is an integral log smooth curve, such that the underlying map of scheme $f\colon C \to X$ is a stable map.

The local structure of $\mathcal{M}_C$ is defined by Fumiharu Kato.

Combinatorial type of a stable log map

Definition

The **combinatorial type** τ of a stable log map $f\colon C/W \to X/S$ consists of:

- The dual intersection graph $G = G_C$ of C, with set of vertices $V(G)$, set of edges $E(G)$, and set of legs $L(G)$.
- The map $\sigma : V(G) \cup E(G) \cup L(G) \to \Sigma(X)$ mapping $x \in C$ (x:generic point (vertex), nodal point (edge), marked point (leg)) to $(\overline{\mathcal{M}}_{X,f(x)})^{\vee}_{\mathbb{R}}$.
- The contact data $u_p \in \overline{M}^{\vee}_{X,f(p)} = \mathrm{Hom}(\overline{M}_{X,f(p)}, \mathbb{N})$ and $u_q \in \mathrm{Hom}(\overline{M}_{X,f(q)}, \mathbb{Z})$ at marked points p and nodes q of C.

Basic monoid

Definition

Given a combinatorial type τ of a stable log map $f\colon C/W \to X/S$, we define the associated **basic monoid** Q by first defining its dual

$$Q_\tau^\vee = \left\{ ((V_\eta)_\eta, (e_q)_q) \in \bigoplus_\eta \overline{M}^\vee_{X,f(\eta)} \oplus \bigoplus_q \mathbb{N} \,\middle|\, \forall q : V_{\eta_2} - V_{\eta_1} = e_q u_q \right\}$$

where the sum is over generic points η of C and nodes q of C. We then set

$$Q_\tau := \mathrm{Hom}(Q_\tau^\vee, \mathbb{N}).$$

- Q_τ indeed only depends on the combinatorial type of $f\colon C/W \to X/S$.
- $Q_{\tau,\mathbb{R}}^\vee := \mathrm{Hom}(Q_\tau, \mathbb{R}_{\geq 0})$ is the moduli cone of tropical curves of fixed combinatorial type.

Basic stable log maps

Given a stable log map $f\colon C/W \to X/S$, one can show that there is a canonical map $Q \to \overline{M}_W$, where Q is the basic monoid defined by the combinatorial type of f.

Definition

A stable log map $f\colon C/W \to X/S$ is said to be **basic** if the natural map of monoids $Q \to \overline{M}_W$ is an isomorphism.

Theorem (Abramovich–Chen, Gross–Siebert, 2011)

The moduli space $\mathcal{M}(X/S)$ of basic stable log maps with target X/S is a Deligne-Mumford stack.

Basic stable log maps

For every $g \in \mathbb{Z}_{\geq 0}$, $\beta \in H_2(X, \mathbb{Z})$ and $u = (u_1, \ldots, u_k)$ with $u_i \in |\Sigma(X)|$, we denote by $\mathcal{M}_{g,u}(X/S, \beta)$ the moduli space of genus g basic stable log maps to X/S of class β and with k marked points of contact data

$$u = u_1, \ldots, u_k.$$

Theorem (Abramovich–Chen, Gross–Siebert, 2011)

- *If X/S is proper, then the moduli space $\mathcal{M}_{g,u}(X/S, \beta)$ is a proper Deligne-Mumford stack.*
- *If X/S is log smooth, then the moduli space $\mathcal{M}_{g,u}(X/S, \beta)$ admits a natural virtual fundamental class $[\mathcal{M}_{g,u}(X/S, \beta)]^{virt}$.*

Counting log maps tropically

Counts of rational log maps in n-dimensional toric varieties

↕

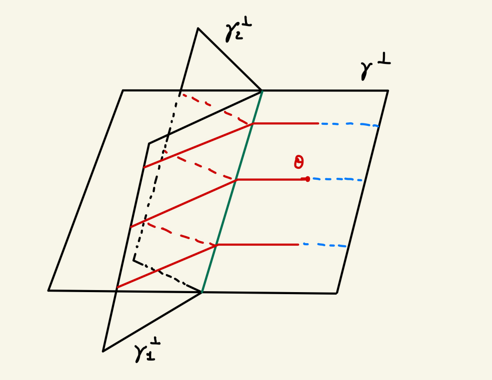
Counts of tropical trees in $\mathbb{R}^n$

- We will work with “families” of tropical trees corresponding to log curves over basic monoids of rank equal to the dimension of the family!

Nishinou–Siebert: Toric degenerations of toric varieties and tropical curves. Duke Mathematical Journal, 2006

Attractor trees to families of tropical curves

- Construction of a $(d-2)$-dimensional family ρ_T of tropical curves in $M_{\mathbb{R}}$ from an attractor tree T:
  - Extend the root of T to infinity to obtain a tropical curve with leaves constrained to lie in the hyperplanes $\gamma_i^{\perp}$.
  - Deform this tropical curve while preserving the combinatorial type and the constraints on the leaves.



The description of $N^{\mathrm{toric}}_{\rho_T,\mathbf{H}}(X_\Sigma)$.

Lemma (A-Bousseau)

For general constraints $\mathbf{H} = (H_1, \ldots, H_r)$*, the number of genus* 0 *log curves in* X_Σ *matching* $\mathbf{H}$*, and with tropicalization the* $(d-2)$*-dimensional family of tropical curves* ρ_T*, denoted by*

$$N^{\mathrm{toric}}_{\rho_T,H}(X_\Sigma)$$

is finite.

- *The proof uses a log generic smoothness (Bertini-Sard) theorem*

Theorem (A-Bousseau)

The coefficients $F^\theta_{r,T}(\gamma_1, \ldots, \gamma_r)$ *expressing the contribution to* $F^\theta_r(\gamma_1, \ldots, \gamma_r)$ *of an attractor tree* T *satisfy*

$$F^\theta_{r,T}(\gamma_1, \ldots, \gamma_r) = N^{\mathrm{toric}}_{\rho_T,H}(X_\Sigma)\,.$$

Summary of the proof

- Construct a toric degeneration $\mathcal{X} \to \mathbb{A}^1$ of X_Σ and of the constraints **H** (similar as in Nishinou-Siebert).
- Degeneration formula: express the invariants $N^{\mathrm{toric}}_{\rho_T,H}(X_\Sigma)$ of the general fibers X_Σ as a sums of invariants $N^{\mathrm{toric}}_{\rho_S}(\mathcal{X}_0)$ of the special fiber $\mathcal{X}_0$, where S are binary trees in $M_\mathbb{R}$ deforming T.
- Show that

$$N^{\mathrm{toric}}_{\rho_S}(\mathcal{X}_0) = \prod_v |\langle \gamma_{v'}, \gamma_{v''} \rangle|$$

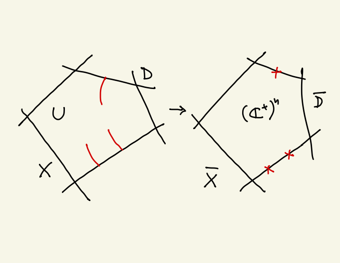
  Key technical point: theory of punctured log maps [Abramovich-Chen-Gross-Siebert] to produce log curves by gluing.
- By the flow tree formula,

$$F^\theta_{r,T}(\gamma_1, \ldots, \gamma_r) = \sum_S \prod_v |\langle \gamma_{v'}, \gamma_{v''} \rangle|$$

A correspondence between quiver DT and log curves in cluster varieties?

Cluster varieties and $\mathbb{A}^1$ curves

- Q, $N = \mathbb{Z}^{Q_0} = \bigoplus_{i \in Q_0} \mathbb{Z} s_i$, $M_{\mathbb{R}} = \mathrm{Hom}(N, \mathbb{R})$, $v_i := \langle s_i, - \rangle \in M$.
  - Fan in $M_{\mathbb{R}}$ containing the rays $\mathbb{R}_{\geq 0} v_i$.
  - Toric variety $\overline{X}$, toric boundary $\overline{D}$, components $(\overline{D}_i)_{i \in Q_0}$.
- X: blow-up of $\overline{X}$ along the codimension two loci $(1 + z^{s_i} = 0)|_{\overline{D}_i}$.
- D: strict transform of $\overline{D}$, (X, D): log Calabi-Yau pair.

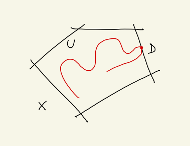


- Complement $U = X \setminus D$, Poisson cluster variety $U = \bigcup (\mathbb{C}^*)^{|Q_0|}$.

M. Gross, P. Hacking and S. Keel, "Birational geometry of cluster algebras," Algebraic Geometry, 2, (2015) 137–175.

Geometry: rational curves in (X, D)

- $\mathbb{A}^1$-curves: rational curves in X meeting D in a single point.



- $\mathbb{A}^1$-curves come in $(d-2)$-dimensional families, where $d = |Q_0| = \dim X$.
- M_β: compactification of the moduli space of $\mathbb{A}^1$-curves of class $\beta \in H_2(X, \mathbb{Z})$.
- GW_β^τ: Counts 0-dimensional strata, "maximally degenerate" $\mathbb{A}^1$-curves.
  - Such counts are punctured log Gromov-Witten invariants of Abramovich–Chen–Gross–Siebert, counting $\mathbb{A}^1$-curves in (X, D) of class β, with degeneration pattern τ. [2]

[2] Argüz–Gross, The Higher Dimensional Tropical Vertex, Geometry & Topology 26 (5), 2135-2235

Quiver-cluster

- Algebra: Ω_γ^θ of the quiver Q are Euler characteristics of moduli spaces of θ-stable representations of Q of dimension γ.
- Geometry: GW_β^τ of the cluster variety (X, D) attached to Q are counts of "maximally degenerate $\mathbb{A}^1$-curves in (X, D) of class β.

Theorem (A-Bousseau)

Assume that the DT attractor invariants of Q are trivial. Then, there exists an explicit correspondence $\beta \to \gamma$, such that

$$\sum_\tau GW_\beta^\tau = \overline{\Omega}_\gamma^\theta .$$

where the sum is over all curves whose tropicalization have type τ, containing one marked leg, tracing out a subspace of $M_\mathbb{R}$ containing θ.

- This is compatible with the previous quiver DT-toric log GW correspondence, the cluster variety (X, D) degenerates to the toric variety $(\overline{X}, \overline{D})$, and the log GW invariants are related.

Heuristic picture of the proof.

- $U = X \setminus D$ admits a Lagrangian torus fibration with base $M_{\mathbb{R}}$.
- Counts GW_{β}^{τ} of $\mathbb{A}^1$-curves in (X, D) are computed by tropical curves in $M_{\mathbb{R}}$.
- $M_{\mathbb{R}}$ is also the space of stability parameters for DT invariants and the same tropical curves describe the wall-crossing behavior of DT invariant DT_{γ}^{θ}!