05.12.2025 : Differential cohomology

<u>Definition</u>: A Cheeger–Simons differential character of degree ℓ is an abelian group morphism
$$\chi \in \mathrm{Hom}(Z_{\ell-1}(M), U(1))$$
such that there exists $F \in \Omega^\ell(M)$, s.t. if $W_{\ell-1} \in Z_{\ell-1}(M)$ satisfies $W_{\ell-1} = \partial B_\ell$ where $B_\ell \in C_\ell(M)$ is a smooth ℓ-chain, then
$$\chi(W_{\ell-1}) = \exp\left(i \int_{B_\ell} F\right).$$

F is called the <u>field strength</u> of χ. Note that if $B_{\ell,1}, B_{\ell,2} \in C_\ell(M)$ s.t. $\partial B_{\ell,1} = \partial B_{\ell,2} = W_{\ell-1}$, then $\exp\left(i \int_{B_{\ell,1} - B_{\ell,2}} F\right) = 1$

This implies that F has periods in $\mathbb{Z}' = 2\pi\mathbb{Z}$, and hence that it is closed.

$$\boxed{\text{The abelian group of CS characters of degree } \ell \text{ is denoted } \check{H}^\ell(M) = \ell\text{-th ordinary diff. cohom. grp of } M.}$$

<u>Example</u>: (Topologically trivial differential characters)

If $A^{(\ell-1)} \in \Omega^{\ell-1}(M)$, on can set
$$\chi_A(W_{\ell-1}) = \exp\left(i \int_{W_{\ell-1}} A\right)$$
$F = dA$

$\Rightarrow$ all periods of F vanish.

Note that for any $\omega^{(\ell-1)} \in \Omega^{(\ell-1)}_{\mathbb{Z}'}(M)$, A and $A+\omega$ define the same character. $\rightsquigarrow$ gauge transformations

<u>Special case</u>: $\omega^{(\ell-1)} = d\lambda^{(\ell-2)}$ where $\lambda^{(\ell-2)} \in \Omega^{(\ell-2)}(M)$

"small gauge transfo" as opposed to "large" ones.

Thus, modulo gauge transformations,
$$\text{top. trivial differential characters} \simeq \Omega^{\ell-1}(M) / \Omega^{\ell-2}_{\mathbb{Z}'}(M)$$

In general, F can have non zero periods

In general, F can have non zero periods. However, locally any F is exact, and hence in the Čech approach one can define differential characters as holonomies.

One speaks of <u>Čech model of differential cohomology</u>. This approach shows that:

① The abelian group morphism
$$\begin{aligned} \check{H}^\ell(M) &\xrightarrow{F} \Omega^\ell_{\mathbb{Z}}(M) \\ (\chi, F) &\longmapsto F \end{aligned}$$
is surjective,

② There is a "characteristic class abelian" group morphism:
$$\begin{aligned} \check{H}^\ell(M) &\xrightarrow{ch} H^\ell(M,\mathbb{Z}) \longrightarrow 0 \\ \chi &\longmapsto ch(\chi) \end{aligned}$$

Moreover, these two maps are compatible in the following sense:

$$\begin{tikzcd}
 & \Omega^\ell_{\mathbb{Z}}(M) \arrow[r] \arrow[dr, "{[\cdot]_{dR}}"] & 0 \\
\check{H}^\ell(M) \arrow[ur, "F"] \arrow[dr, "ch"'] \arrow[rr, phantom, "\circlearrowright"] & & H^\ell(M,\mathbb{R}) \\
 & H^\ell(M,\mathbb{Z}) \arrow[ur, "\otimes\mathbb{R}"'] \arrow[dr] & \\
 & & 0
\end{tikzcd}$$

We are now going to complete this diagram into a hexagon, characteristic of differential cohomology.

<u>Definition</u>: Differential characters in the kernel of F are said to be <u>flat</u>, those in the kernel of ch are said to be <u>topologically trivial</u>.

We have already remarked that topologically trivial characters correspond to $\Omega^{\ell-1}(M)/\Omega^{\ell-1}_{\mathbb{Z}}(M)$:

$$0 \to \Omega^{\ell-1}(M)/\Omega^{\ell-1}_{\mathbb{Z}}(M) \longrightarrow \check{H}^\ell(M) \longrightarrow H^\ell(M,\mathbb{Z}) \to 0.$$

Flat differential characters: By definition, a flat differential character

$$\chi : Z_{\ell-1}(M) \longrightarrow U(1)$$

factorizes through $H_{\ell-1}(M, \mathbb{Z})$, i.e. defines

$$Z_{\ell-1}(M) \twoheadrightarrow H_{\ell-1}(M,\mathbb{Z}) \xrightarrow{\chi} U(1).$$

Facts: $H^{\ell-1}(M, U(1)) \simeq \mathrm{Hom}(H_{\ell-1}(M), U(1))$.

$$\begin{gathered} \| \\ H^{\ell-1}(M, \mathbb{R}/\mathbb{Z}') \end{gathered}$$

If M is compact, which we assume, then $H^{\ell-1}(M, \mathbb{R}/\mathbb{Z}')$ is a compact abelian group.

Any compact abelian group A fits in a ses

$$0 \longrightarrow A_0 \longrightarrow A \longrightarrow \pi_0(A) \longrightarrow 0,$$

where A_0 is the connected component of $id \in A$, and where $\pi_0(A)$ is a finite abelian group, i.e.

$$\pi_0(A) \simeq \bigoplus_i \mathbb{Z}_{n_i}$$

Examples: 1) $A = (\mathbb{R}/\mathbb{Z})^b \oplus \bigoplus_i \mathbb{Z}_{n_i}$

2) $1 \longrightarrow SO(2) \longrightarrow O(2) \longrightarrow \mathbb{Z}/_{2\mathbb{Z}} \longrightarrow 1$

For $A = H^{\ell-1}(M, \mathbb{R}/\mathbb{Z}')$, $A_0 = H^{\ell-1}(M, \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}'$
(connected) abelian group of Wilson lines.

The ses $0 \to \mathbb{Z}' \to \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}' \longrightarrow 0$ gives rise to a long exact sequence

$$\cdots \to H^{\ell-1}(M, \mathbb{R}) \longrightarrow H^{\ell-1}(M, \mathbb{R}/\mathbb{Z}') \xrightarrow[\beta]{} H^{\ell}(M, \mathbb{Z}') \to H^{\ell}(M, \mathbb{R}) \to \cdots$$

The connecting map β is called the Bockstein map.

By exactness, $\mathrm{im}(\beta) = \ker(H^\ell(M, \mathbb{Z}') \longrightarrow H^\ell(M, \mathbb{R}))$
$= \mathrm{Tor}(H^\ell(M, \mathbb{Z}'))$,

By exactness, $\mathrm{im}(\beta) = \ker\left(H^{\ell}(M,\mathbb{Z}') \longrightarrow H^{\ell}(M,\mathbb{R})\right)$
$= \mathrm{Tor}\left(H^{\ell}(M,\mathbb{Z}')\right)$,
where if A is a finitely generated abelian group, $\mathrm{Tors}(A)$ is the finite abelian subgroup of A fitting in

$$0 \longrightarrow \mathrm{Tors}(A) \longrightarrow A \longrightarrow \overline{A} \longrightarrow 0$$

where $\overline{A} \simeq \mathbb{Z}^b$, and $A \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^b$.

<u>Thus</u>: $\pi_0\left(H^{\ell-1}(M,\mathbb{R}/\mathbb{Z}')\right) \simeq \mathrm{Tors}\left(H^{\ell}(M,\mathbb{Z}')\right)$.

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& \dfrac{\Omega^{\ell-1}(M)}{\Omega^{\ell-1}_{\mathbb{Z}'}(M)} \arrow[rr, "d"] \arrow[dr] & & \Omega^{\ell}_{\mathbb{Z}'}(M) \arrow[ur] \arrow[dr, "{[\cdot]_{dR}}"] & \\
H^{\ell-1}(M,\mathbb{R}) \arrow[ur] \arrow[dr] & & \check{H}^{\ell}(M) \arrow[ur, "F"] \arrow[dr, "ch"] & & H^{\ell}(M,\mathbb{R}) \\
& H^{\ell-1}(M,\mathbb{R}/\mathbb{Z}') \arrow[rr, "\beta"'] \arrow[ur] & & H^{\ell}(M,\mathbb{Z}) \arrow[ur, "\otimes\mathbb{R}"'] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

<u>Hopkins–Singer differential cochains</u>: (0211216)

Consider an $(\ell-1)$-cycle Σ on M which is n-torsion, i.e.

$$n\Sigma = \partial B \quad \text{where } B \in C_n(M,\mathbb{Z})$$

integral n chain

$$\left[\; \underset{\text{cycles}}{Z_{\ell}(M,\mathbb{Z})} \subset \underset{\text{chains}}{C_{\ell}(M,\mathbb{Z})} \qquad Z_{\ell}(M,\mathbb{Z}) = \ker\left(C_{\ell}(M,\mathbb{Z}) \xrightarrow{\;d\;} C_{\ell-1}(M,\mathbb{Z})\right) \right]$$

Then $\chi(n\Sigma) = \chi(\Sigma)^n = \exp\left(i\int_B F\right)$

<u>Remark</u>: This does NOT imply that $\chi(\Sigma) \overset{(*)}{=} \exp\left(\frac{i}{n}\int_B F\right)$
As $F \in \Omega^{\ell}_{\mathbb{Z}'}(M)$ and not necessarily
$F \in \Omega^{\ell}_{n\mathbb{Z}'}(M)$
$(*)$ is ill-defined?

<u>BUT</u>: $\exists\, a \in C^{\ell}(M,\mathbb{Z})$ s.t.

BUT: $\exists\, a \in C^{\ell}(M,\mathbb{Z})$ s.t. *

$$\chi(\Sigma) = \exp\left\{ \frac{i}{n}\left(\int_B F - 2\pi \langle a, B\rangle\right)\right\} \text{ is well-defined.}$$

One has $\delta a = 0 \Rightarrow [a] \in H^{\ell}(M,\mathbb{Z})$

(in fact $[a] = cl(\chi)$).

Heuristically: "$\delta \log \chi \sim F - a$".

More precisely: ① Extend χ to $(\ell-1)$-chains which are not necessarily cycles (arbitrarily)

i.e. $\chi : C_{\ell-1}(X,\mathbb{Z}) \longrightarrow U(1)$

$\chi \in C^{\ell-1}(X, U(1))$

Let $A \sim \frac{1}{i}\log\chi$ be an arbitrary logarithm of χ,

i.e. $A \in C^{\ell-1}(X,\mathbb{R})$

$$M \longmapsto \int_M A \in \mathbb{R}$$

Coboundary: $\delta A \in C^{\ell}(X,\mathbb{R})$: $\int_N \delta A = \int_{\partial N} A$

Now, $\exp\left(i\int_N F\right) = \chi(\partial N) = \exp\left(i\int_{\partial N} A\right)$

$$\Rightarrow \int_N (F - \delta A) \in \mathbb{Z}'$$

Thus, $\exists\, c \in C^{\ell}(X,\mathbb{Z}')$ s.t. $\delta A = F - c$

Definition: A Hopkins–Singer cocycle is a triple $(c, A, F) \in C^{\ell}(M,\mathbb{Z}') \times C^{\ell-1}(M,\mathbb{R}) \times \Omega^{\ell}(M)$ such that $\delta A = F - c$.

"Gauge" redundancies: ① Extension of χ:

$$A \longrightarrow A + \delta a$$

where $a \in C^{\ell-2}(M,\mathbb{R})$

② Logarithm: $A \longrightarrow A + n$ where $n \in C^{\ell-1}(M,\mathbb{Z}')$

$\rightsquigarrow c \longrightarrow c - \delta n$

$\rightsquigarrow \quad c \longrightarrow c - \delta n$

$$\boxed{(c, A, F) \longrightarrow (c - \delta n,\ A + \delta a + n,\ F)}$$

where $a \in C^{\ell-2}(M, \mathbb{R})$, $n \in C^{\ell-1}(M, \mathbb{Z}')$.

$\Rightarrow$ Chain complex (cf. 02 11 2016)

$$\check{C}(q)^n(M) = \begin{cases} C^n(M, \mathbb{Z}') \times C^{n-1}(M, \mathbb{R}) \times \Omega^n(M) & (n \geq q) \\ C^n(M, \mathbb{Z}') \times C^{n-1}(M, \mathbb{R}) & (n < q) \end{cases}$$

$$d(c, h, \omega) = (\delta c,\ \omega - c - \delta h,\ d\omega)$$

$$d(c, h) = \begin{cases} (\delta c, -c - \delta h, 0) & (c, h) \in \check{C}(q)^{q-1} \\ (\delta c, -c - \delta h) & \text{otherwise} \end{cases}$$

Fact: $\boxed{\check{H}(q)^q(M) = \check{H}^{q-1}(M)}$

Properties of differential cohomology:

Ring structure: $\check{H}^{\ell_1} \times \check{H}^{\ell_2} \longrightarrow \check{H}^{\ell_1 + \ell_2}$

$$x_1 \cdot x_2 = (-1)^{\ell_1 \ell_2} x_2 \cdot x_1$$

AND $F(x_1 \cdot x_2) = F(x_1) \wedge F(x_2)$

$c(x_1 \cdot x_2) = c(x_1) \cup c(x_2)$

Holonomy: $(c_1, A_1, F_1) \cdot (c_2, A_2, F_2)$
$= (c_1 \cup c_2,\ \pm c_1 \cup A_2 + A_1 \cup c_2 + H(F_1, F_2),$
$F_1 \wedge F_2)$

w/ H a homotopy btw cup product $\cup$ and wedge product $\wedge$ on Ω^ℓ (smooth $\mathbb{R}$-valued ℓ cochains).

Integration: Let $\Sigma_\ell \in Z_\ell(M, \mathbb{Z})$

$$\int_{\Sigma_\ell} : \check{H}^{\ell+1}(\Sigma_\ell) \longrightarrow \check{H}^1(pt) = \mathbb{R}/\mathbb{Z}$$

Generalizes to families:
$$\begin{array}{ccc} M^{(n)} & \longrightarrow & \mathfrak{X} \\ \text{(closed } n\text{-manifolds)} & & \downarrow \\ & & S \end{array}$$

$$\int_{\mathfrak{X}/S} : \check{H}^d(\mathfrak{X}) \longrightarrow \check{H}^{d-n}(S)$$

Pairing = multiply and integrate

$$\check{H}^{\ell}(M^{(n)}) \times \check{H}^{n-\ell+1}(M^{(n)}) \longrightarrow \check{H}^1(pt) = \mathbb{R}/\mathbb{Z}$$

$$\langle [\check{A}_1], [\check{A}_2] \rangle = \int_{M^{(n)}} [\check{A}_1] \cdot [\check{A}_2] \in \mathbb{R}/\mathbb{Z}$$

$$\left[\underline{n=3} : \check{H}^2(M^{(3)}) \times \check{H}^2(M^{(3)}) \longrightarrow \mathbb{R}/\mathbb{Z} \right]$$

Special case: When $[\check{A}_1]$ is topologically trivial,
i.e. $F_1 = dA_1$ for $A_1 \in \Omega^{\ell-1}(M)$
then $\langle [\check{A}_1], [\check{A}_2] \rangle = \int_{M_n} A_1 F_2$.

The pairing in differential cohomology provides a good definition of BF and Chern–Simons actions.

Remark: In fact $\check{H}^{\ell}(M_n) \times \check{H}^{n-\ell+1}(M_n) \longrightarrow \mathbb{R}/\mathbb{Z}$
is a perfect pairing known as
Poincaré – Pontryagin duality of differential cohomology.

$(X, A) = i : A \hookrightarrow X$
$(X, A) \longrightarrow (X', A')$
$\iff f : X \to X'$
$g : A \to A'$
s.t. $i' \circ g = f \circ i$

Generalized cohomology theories:

Cohomology (Eilenberg – Steenrod axioms)
Sequence of functors from the category of pairs of topological spaces (X, Y) to the category of abelian groups, together w/ a natural transformation
$$\delta : H^k(Y) \longrightarrow H^{k+1}(X, Y) \qquad \left(H^k(Y) = H^k(Y, \emptyset) \right)$$
(coboundary map)
satisfying the following axioms:

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① Homotopy invariance

if $g:(X,Y) \longrightarrow (X',Y')$ is homotopic to $h:(X,Y) \longrightarrow (X',Y')$

then their induced homomorphisms are the same

② Exactness

$$Y \overset{i}{\hookrightarrow} X \overset{j}{\hookrightarrow} (X,Y)$$

$$\cdots H^k(X,Y) \xrightarrow{j^*} H^k(X) \xrightarrow{i^*} H^k(Y) \xrightarrow{\delta^*} H^{k+1}(X,Y) \rightarrow \cdots$$

③ Excision

If $U \subset Y$ s.t. $\overline{U} \subset \mathring{Y}$ then

$i:(X\setminus U, Y\setminus U) \longrightarrow (X,Y)$

induces an isomorphism $H^k(X,Y) \simeq H^k(X\setminus U, Y\setminus U)$

④ Additivity

$$H^k\left(\coprod_\alpha X_\alpha\right) \simeq \bigoplus_\alpha H^k(X_\alpha)$$

⑤ Point axiom

$$H^k(pt,\phi) = \begin{cases} \mathbb{Z} & k=0 \\ 0 & k\neq 0 \end{cases}$$

① – ④ $\Rightarrow$ Mayer-Vietoris

Generalized cohomology theory: ① – ④ but not ⑤

Ex: complex K-theory $K^j(pt) = \begin{cases} \mathbb{Z} & j \text{ even} \\ 0 & j \text{ odd} \end{cases}$

real, $\mathbb{H}$, ... K-theory, bordisms,

cobordisms ...

Hopkins-Singer $\Rightarrow$ to any GCT there is a differential version satisfying a hexagon diagram

Spectrum: An object representing a GCT

($\exists$ follows from Brown representability thm)

$\forall \quad \mathcal{E}^*: CW^{op} \longrightarrow Ab \quad$ GCT

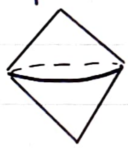
$\exists$ spaces E^k s.t. evaluating the theory in degree k on X is equivalent to computing the homotopy classes of maps to E^k, i.e.

$$\mathcal{E}^k(X) \simeq [X, E^k]$$

A (CW)-spectrum is a sequence $E := \{E_n\}_{n \in \mathbb{N}}$ of CW-complexes together with inclusions

$$\Sigma E_n \longrightarrow E_{n+1}$$

(suspension)



$X \; ; \; \Sigma X = X \times [0,1] \Big/ \begin{smallmatrix} X \times \{0\} \\ X \times \{1\} \end{smallmatrix}$

$$\mathrm{Maps}_*(\Sigma X, Y) \simeq \mathrm{Maps}_*(X, \Omega Y)$$

<u>Eilenberg-Maclane spectrum</u> $\quad A$ abelian group

$$H^n(X; A) = [X, K(A, n)]$$

<u>Complex K-theory</u>: $E^{2n} = \mathbb{Z} \times BU \longrightarrow$ classifying space of infinite unitary group

$E^{2n+1} = U$