03.12.2025 **Towards differential cohomology**

Spacetime M a smooth d-dimensional manifold with the homotopy type of a CW-complex.
Then:

$$\boxed{H^2(X,\mathbb{Z}) \simeq \{L \to X : \text{Hermitian LB}\}/\sim \quad \text{(first Chern class)}}$$

Take a $U(1)$ connection ∇ on L (locally $\nabla = d + iA$)

$\rightsquigarrow$ curvature $F_\nabla = \nabla^2 \in \Omega^2_c(X;\mathbb{R})$ (closed forms).

$$\underset{c_1(L)\otimes_{\mathbb{Z}}\mathbb{R}}{\overset{\shortparallel}{c_1(L)_{\mathbb{R}}}} = c_1(F_\nabla) := \frac{1}{2\pi}[F_\nabla] \in H^2_{dR}(X) \simeq H^2(X,\mathbb{R})$$

The curvature recovers $c_1(L)$ up to torsion.

$(c_1(F_\nabla) \in \Omega^2_c(X)_{\mathbb{Z}'})$

$\hookrightarrow$ periods in $\mathbb{Z}' = 2\pi\mathbb{Z}$.

Flat line bundles

(L,∇) s.t. $F_\nabla = 0$

HOWEVER: Not all flat line bundles are trivial!

For instance, when $H^2(X,\mathbb{Z})$ is torsion, e.g. $\mathbb{Z}_2$, then $c_1(L)_{\mathbb{R}} = 0$ $\forall L$.

Ex: $X = \mathbb{RP}^2 = S^2/\mathbb{Z}_2$

Trivial line bundle $\mathbb{C}\times S^2 \longrightarrow S^2$, $\nabla = d$

$(z,x) \longmapsto (-z,-x)$ preserves d!

Quotient $\rightsquigarrow L \longrightarrow \mathbb{RP}^2$ with ∇

L is not trivial: $c_1(L) = -1 \in H^2(\mathbb{RP}^2,\mathbb{Z}) = \mathbb{Z}_2$

Holonomy: $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$

$\mathrm{Hol}(L,\nabla) \in \mathrm{Hom}(\pi_1(\mathbb{RP}^2), U(1)) \simeq \mathbb{Z}_2$

$(= \widehat{\mathbb{Z}_2})$.

$\rightsquigarrow$ gauge invariant of (L,∇)!

$\mathrm{Hol}(L,\nabla) : \mathcal{E}^\infty(S^1, X) \longrightarrow U(1) = \mathbb{R}/\mathbb{Z}$

Holonomy detects the non-triviality of flat line bundles.

(Higher form) Gauge fields

For instance, p-form Maxwell: ($M^{(d)}$ pseudo Riemannian)

$$S[\check{A}^{(p)}] = \int_{M^{(d)}} F^{(p+1)} \wedge (\ast F)^{(d-p-1)}$$

p-form gauge field

, $p < d-1$

$F^{(p+1)} \in \Omega^{(p+1)}_{\mathbb{Z}'}(M)$ field strength of $\check{A}^p$

where $\Omega^{(p+1)}_{\mathbb{Z}'}(M) =$ closed $(p+1)$ differential forms on M with periods valued in $\mathbb{Z}' := 2\pi\mathbb{Z}$.

Other examples?

* B-field in string theory
* BF-theories } Definition of the action requires care
* CS-theories }
* M5-brane gauge field (chiral ~ self dual)
* RR fields (K-theoretic)

There is no satisfying geometric model generalizing unitary connections on hermitian line bundles...

BUT: the local (Čech) model generalizes well.

$p=1$: Any $U(1)$-connection is locally trivial, i.e. $\exists$ a good open cover $\mathcal{U} = (\mathcal{U}_\alpha)_{\alpha \in I}$ of M s.t.

$\forall \alpha$, on $\mathcal{U}_\alpha$ $\quad \check{A}^{(1)} : \quad d + A^{(1)}_\alpha \quad , \quad A_\alpha \in \Omega^1(\mathcal{U}_\alpha)$

$\forall \alpha, \beta$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$: $\quad A^{(1)}_\alpha - A^{(1)}_\beta = d f^{(0)}_{\alpha\beta} \quad (\ast)$

where $f^{(0)}_{\alpha\beta} \in \Omega^0(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$

$\forall \alpha, \beta, \gamma$, on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma =: \mathcal{U}_{\alpha\beta\gamma}$

$$d(f^{(0)}_{\alpha\beta} + f^{(0)}_{\beta\gamma} + f^{(0)}_{\gamma\alpha}) = 0$$

$$\Rightarrow f^{(0)}_{\alpha\beta} + f^{(0)}_{\beta\gamma} + f^{(0)}_{\gamma\alpha} = \text{constant}$$

Denoting $g_{\alpha\beta} = e^{i f^{(0)}_{\alpha\beta}}$ ($U(1)$ gauge transformation)

$$\Rightarrow d f^{(0)}_{\alpha\beta} = -i g_{\alpha\beta}^{-1} d g_{\alpha\beta}$$

$\Rightarrow g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha}$ constant on $\mathcal{U}_{\alpha\beta\gamma}$

We need $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$.

Note that this implies $f^{(0)}_{\alpha\beta} + f^{(0)}_{\beta\gamma} + f^{(0)}_{\gamma\alpha} = 2\pi n_{\alpha\beta\gamma}$
where $n_{\alpha\beta\gamma} \in \mathbb{Z}$.
On $\mathcal{U}_{\alpha\beta\gamma\delta}$: $n_{\alpha\beta\gamma} - n_{\alpha\beta\delta} + n_{\alpha\gamma\delta} - n_{\beta\gamma\delta} = 0$

$\Rightarrow \{n_{\alpha\beta\gamma}\}_{\mathcal{U}_{\alpha\beta\gamma}}$ is a Čech 2-cocycle with cohomology class in $\check{H}^2(\mathcal{U}, \mathbb{Z}) \simeq H^2(M, \mathbb{Z}) \ni c_1$
(first Chern class).

Remark: $\circledast \Rightarrow F^{(2)} = dA^{(1)}$ is globally well-defined
Moreover the de Rham class of $F^{(2)}$ is
$c_{1,\mathbb{R}} \in H^2(M, \mathbb{R})$ i.e. $[F^{(2)}] = c_1 \otimes \mathbb{R}$.
Reciprocally, every $F^{(2)} \in \Omega^{(2)}_c(M)$ is locally exact ?

p = 0: Periodic scalar $f: M \longrightarrow U(1)$
$\exists\ \mathcal{U} = (\mathcal{U}_\alpha)_\alpha$ s.t. on $\mathcal{U}_\alpha$, f has a well defined logarithm, i.e. $f_\alpha = e^{i\phi_\alpha}$
On $\mathcal{U}_\alpha$, $\phi_\alpha \in \Omega^0(\mathcal{U}_\alpha)$
$\mathcal{U}_{\alpha\beta}$: $\phi_\alpha - \phi_\beta = 2\pi n_{\alpha\beta}$
[Note that $F^{(1)} = d\phi$ is globally well defined]
$\mathcal{U}_{\alpha\beta\gamma}$: $\phi_\alpha - \phi_\beta + \phi_\beta - \phi_\gamma + \phi_\gamma - \phi_\alpha = 0$
$= 2\pi (n_{\alpha\beta} + n_{\beta\gamma} + n_{\gamma\alpha})$
$\Rightarrow \{n_{\alpha\beta}\}$ is a Čech 1-cocycle and defines a cohomology class c in $H^1(M, \mathbb{Z})$.
$[F^{(1)}]_{dR} = c \otimes \mathbb{R} \in H^1(M, \mathbb{R})$.

p = 2: Abelian gerbe connection
Each $H^{(3)} \in \Omega^3_c(M)$ is locally exact
$\mathcal{U}_\alpha$: $H^{(3)}|_{\mathcal{U}_\alpha} = dB^{(2)}_\alpha$, $B^{(2)}_\alpha \in \Omega^2(M)$
$\mathcal{U}_{\alpha\beta}$ $B^{(2)}_\alpha - B^{(2)}_\beta = d\lambda^{(1)}_{\alpha\beta}$
$\mathcal{U}_{\alpha\beta\gamma}$ $\lambda^{(1)}_{\alpha\beta} + \lambda^{(1)}_{\beta\gamma} + \lambda^{(1)}_{\gamma\alpha} = df^{(0)}_{\alpha\beta\gamma}$

Let $df^{(0)}_{\alpha\beta\gamma} = -i g^{-1}_{\alpha\beta\gamma}\, d g_{\alpha\beta\gamma}$ $\quad (U(1))$
and impose:

$$[F^{(1)}]_{dR} = c \otimes \mathbb{R} \in H^1(M,\mathbb{R}).$$

$\underline{p=2}$: Abelian gerbe connection

Each $H^{(3)} \in \Omega^3_c(M)$ is locally exact

$$\mathcal{U}_\alpha : \quad H|_{\mathcal{U}_\alpha} = dB^{(2)}_\alpha, \quad B^{(2)}_\alpha \in \Omega^2(M)$$

$$\mathcal{U}_{\alpha\beta} \quad B^{(2)}_\alpha - B^{(2)}_\beta = d\lambda^{(1)}_{\alpha\beta}$$

$$\mathcal{U}_{\alpha\beta\gamma} \quad \lambda^{(1)}_{\alpha\beta} + \lambda^{(1)}_{\beta\gamma} + \lambda^{(1)}_{\gamma\alpha} = df^{(0)}_{\alpha\beta\gamma}$$

Let $df^{(0)}_{\alpha\beta\gamma} = -i g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma}$ $\quad (U(1))$

and impose:

$$g_{\alpha\beta\gamma}\, g^{-1}_{\alpha\gamma\delta}\, g_{\alpha\beta\delta}\, g^{-1}_{\beta\gamma\delta} \text{ on } \mathcal{U}_{\alpha\beta\gamma\delta}$$

$$\Rightarrow f_{\alpha\beta\gamma} - f_{\alpha\gamma\delta} + f_{\alpha\beta\delta} - f_{\beta\gamma\delta} = 2\pi m_{\alpha\beta\gamma\delta}$$

where $m_{\alpha\beta\gamma\delta} \in \mathbb{Z}$

On $\mathcal{U}_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}$, $\sum_{i=1}^{5} (-1)^i\, m_{\alpha_1 \cdots \widehat{\alpha_i} \cdots \alpha_5} = 0$

$\rightsquigarrow \{m_{\alpha\beta\gamma\delta}\}$ is a Čech 3-cocycle and defines a class c in $H^3(M,\mathbb{Z})$.

Moreover $[H^{(3)}]_{dR} = c \otimes \mathbb{R} \in H^3(M,\mathbb{R})$.

The construction generalizes to any p

Recursively: a $U(1)$ p-form gauge field is defined on a good cover $\mathcal{U} = \{\mathcal{U}_\alpha\}$ to be the data of a differential p-form $A^{(p)}_\alpha$ on each $\mathcal{U}_\alpha$, such that $\forall \alpha, \beta$, on $\mathcal{U}_{\alpha\beta}$ $A^{(p)}_\alpha - A^{(p)}_\beta = d\lambda^{(p-1)}_{\alpha\beta}$ where $\lambda^{(p-1)}_{\alpha\beta}$ is a $U(1)$ $(p-1)$-form gauge field on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$

Note: ① $F^{(p+1)} = dA^{(p)} \in \Omega^{p+1}(M)$ well-defined,

② $\check{A}^{(p)}$ defines a Čech $(p+1)$-cocycle, and in turn, a cohomology class $c \in H^{(p+1)}(M,\mathbb{Z})$ called the characteristic class,

③ $[F^{(p+1)}]_{dR} = c \otimes \mathbb{R}$.

④ $\forall$ smooth p-cycle $X_{(p)} \in Z_p(M,\mathbb{Z})$, one can define the holonomy of $\check{A}^{(p)}$:

$$\mathrm{Hol}(\check{A}^{(p)}, X_{(p)}) \in U(1).$$