28.11.2025

Magnetic monopoles and Cech description of gauge fields

Goal: Discuss subtleties with gauge fields.
We will focus on the abelian ($U(1)$) case.

What is the gauge-invariant info. carried by gauge fields?

What are higher-form gauge fields?

Start with pure Maxwell on some Lorentzian (paracompact)

$(\hat{r}, \hat{\theta}, \hat{\varphi})$

$$S = \frac{-1}{4e^2}\int_M d^4x \, F_{\mu\nu}F^{\mu\nu} = \frac{-1}{2e^2}\int_M F \wedge \star F$$

S^2
(fixed time)

$$\vec{E} = \frac{q}{4\pi r^2}\hat{r} \qquad \text{Gauss law} \int_{S^2} \vec{E}\cdot d\vec{S} =$$

$$\vec{B} = \frac{g}{4\pi r^2}\hat{r}$$

Magnetic Gauss law: $\displaystyle\int_{S^2} \vec{B}\cdot d\vec{S} = g$

How is this possible?

$$dF = 0 \Leftrightarrow \vec{\nabla}\cdot\vec{B} = 0 \;\&\; \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}:$$

$$\Downarrow$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

Loophole: $\vec{B} = \vec{\nabla}\times\vec{A}$ would imply $\int_{S^2}\vec{B}\cdot d\vec{S} = 0$
if $\vec{A}$ were a globaly-define 1-form on M.
BUT $\vec{A} \notin \Omega^1(M, \mathbb{R})$ in general.
Rather, $\vec{A}$ is a $\boxed{U(1) \text{ connection 1-form}}$, and it can be represented by 1-form only locally.

<u>Definition</u>: Let $\{U_\alpha\}_{\alpha\in A}$ be an open covering of M
$\forall \alpha,\beta\in A$, transition functions
$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(m,k),\ k=\mathbb{R} \text{ or } \mathbb{C}$$
s.t $\forall \alpha,\beta,\gamma\in A$
$$g_{\alpha\beta}\, g_{\beta\gamma} = g_{\alpha\gamma} \text{ on } U_\alpha\cap U_\beta\cap U_\gamma$$
cocycle condition.

Let $\tilde{E} = \{(\alpha,p,v)\in A\times M\times k^m\}$
Equivalence relation $\sim$ on $\tilde{E}$:
$$(\alpha,p,v)\sim(\beta,q,w) \iff \begin{array}{l} p=q\in U_\alpha\cap U_\beta \\ v = g_{\alpha\beta}(p)w \end{array}$$

$[\alpha,p,v]$ equivalence class
$E = \tilde{E}/\sim$ set of equivalence classes

$$\pi : E \longrightarrow M \qquad [\alpha,p,v]\longmapsto p$$

$$\begin{array}{ccc} \tilde{U}_\alpha = \pi^{-1}(U_\alpha) & \ni & [\alpha,p,v] \\ \downarrow \psi_\alpha & & \downarrow \\ U_\alpha\times k^m & & (p,v) \end{array}$$

$\exists!$ manifold structure on E making each ψ_α diffeos

<u>$k=\mathbb{R}$</u>: real vector bundle of rank m
<u>$k=\mathbb{C}$</u>: complex v.b of rank m

In what follows, consider $k=\mathbb{C}$, $m=1$
$\longrightarrow$ complex line bundle.

Notion of section, iso, etc...

<u>Example</u>: $M = S^2 = \mathbb{C}\cup\{\infty\}\ni z$
$$U_0 = S^2 - \{\infty\} \qquad U_\infty = S^2-\{0\}$$
$$g_{\infty 0} : U_\infty\cap U_0 \longrightarrow GL(1,\mathbb{C})$$
$$z \longmapsto \frac{1}{z^n}$$
$\Rightarrow$ complex line bundle H^n

<u>Fact</u>: Any complex line bundle over S^2 is isomorphic to some H^n for $n\in\mathbb{Z}$.

<u>More generally</u>
$$\boxed{\text{If } M \text{ is a smooth manifold, } \mathrm{Vect}_1^{\mathbb{C}}(M) \simeq H^2(M,\mathbb{Z})}$$

G-vector bundle: v.b. w/ transition functions valued in G.

$G = U(m) \Rightarrow$ transition functions preserve the usual hermitian metric on $\mathbb{C}^m$

$\Leftrightarrow$ $U(m)$-vector bundle is a rank-m complex vector bundle together with a hermitian metric.

We are interested in $U(1)$ line bundles.

Definition: A connection on a vector bundle E is a map $d_A : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$ s.t. $d_A(f\sigma + \tau) = (df) \otimes \sigma + f d_A\sigma + d_A\tau$

Locally (e.g. on $\{U_\alpha\}$) there exist matrices of 1-form $\omega_\alpha \in \Omega^1(U_\alpha, M_m(k))$

$$d_A = d + \omega_\alpha$$

$$d_A \Leftrightarrow \{d + \omega_\alpha\}_{\alpha \in A}$$

$$\omega_\alpha = g_{\alpha\beta}\, d g_{\alpha\beta}^{-1} + g_{\alpha\beta}\, \omega_\beta\, g_{\alpha\beta}^{-1} \quad \text{on } U_\alpha \cap U_\beta$$

Unitary connection: ω_α takes values in $\underline{u}(m)$

$\underline{u}(m) = \{ m \times m$ skew-Hermitian matrices $M = -M^\dagger \}$

$k = \mathbb{C}$, $m = 1$, unitary connection:

$$\omega_\alpha \in \Omega^1(U_\alpha, i\mathbb{R})$$

$$\omega_\alpha = i A_\alpha$$

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow U(1)$$

If $U_\alpha \cap U_\beta$ is simply connected:

$g_{\alpha\beta} = e^{i\vartheta_{\alpha\beta}}$ where $\vartheta_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathbb{R}$

$$g_{\alpha\beta}\, d g_{\alpha\beta}^{-1} = -e^{i\vartheta_{\alpha\beta}}\, i d\vartheta_{\alpha\beta}\, e^{-i\vartheta_{\alpha\beta}} = -i d\vartheta_{\alpha\beta}$$

$$\omega_\alpha = \omega_\beta - i d\vartheta_{\alpha\beta}$$

$$\Rightarrow \boxed{A_\alpha = A_\beta - d\vartheta_{\alpha\beta}}$$

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> The gauge field of electromagnetism is a $U(1)$-connection on a complex line bundle over space-time.

Back to the monopole:

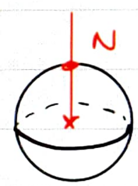
$\vec{A}$ not defined at the origin of $\mathbb{R}^3 \longrightarrow \mathbb{R}^3 \setminus \{0\}$.

FINE

$$A_\phi^N = \frac{g}{4\pi r} \cdot \frac{1-\cos\vartheta}{\sin\vartheta}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(A_\phi^N \sin\vartheta\right)\vec{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(r A_\phi^N\right)\vec{\vartheta}$$

$$\Rightarrow \vec{B} = \frac{g\vec{r}}{4\pi r^2}$$



$$A_\phi^S = -\frac{g}{4\pi r}\,\frac{1+\cos\vartheta}{\sin\vartheta}$$

$$\vec{A} \longrightarrow \vec{A} + \vec{\nabla}\omega \; ; \quad A_\phi^N = A_\phi^S + \frac{1}{r\sin\vartheta}\partial_\phi \omega \quad (*)$$

$(\vartheta \neq 0, \pi)$

where $\omega = \dfrac{g\phi}{2\pi}$

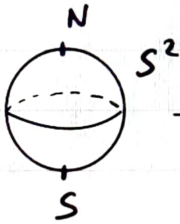
However, note that

$$\omega(\phi = 2\pi) = \omega(\phi = 0) + g \,!$$

> Spherical coordinates (r, ϑ, φ):
>
> $$\vec{\nabla} f = \frac{\partial f}{\partial r}\vec{r} + \frac{1}{r}\frac{\partial f}{\partial\vartheta}\vec{\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial f}{\partial\varphi}\vec{\varphi}$$
>
> $$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta A_\vartheta\right) + \frac{1}{r\sin\vartheta}\frac{\partial A_\varphi}{\partial\varphi}$$
>
> $$\vec{\nabla}\times\vec{A} = \frac{1}{r\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}\left(A_\varphi \sin\vartheta\right) - \frac{\partial A_\vartheta}{\partial\varphi}\right]\vec{r} + \frac{1}{r}\left[\frac{1}{\sin\vartheta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(r A_\varphi\right)\right]\vec{\vartheta} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r A_\vartheta\right) - \frac{\partial A_r}{\partial\vartheta}\right]\vec{\varphi}$$

Gauge transformation? $\psi \to e^{iew/\hbar}\psi$ (QM) *

Require that ψ is single valued; this holds for (*) provided $eg = 2\pi\hbar n$, $n \in \mathbb{Z}$!
(w does not have to be single valued, only ψ does).



$\to U^N = S^2 - \{S\}$; $U^S = S^2 - \{N\}$

"Clutching construction"

What are we doing?

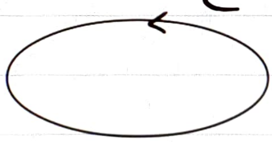
$\Rightarrow$ non-trivial $U(1)$ bundle over S^2;
$n = c_1$ = first Chern class.

Moral of the story:
Gauge fields and gauge parameters are differential forms only locally.

Dirac quantization condition

Background: $\vec{A}(x,t)$

$\boxed{\text{QM}}$

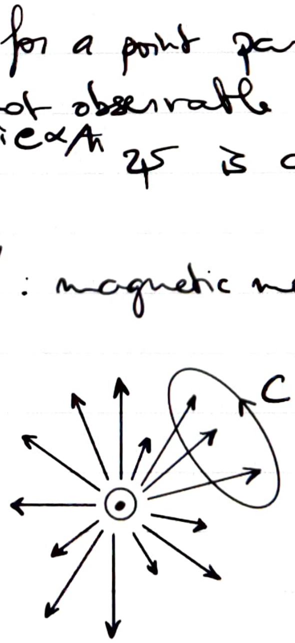


C closed path

$$\psi \longrightarrow e^{ie\alpha/\hbar}\psi \qquad \alpha = \oint_C \vec{A}\cdot d\vec{x}$$

(action for a point particle include $\int dt \, e\dot{\vec{x}}\cdot\vec{A}$)
Phase is not observable per se in QM, but $e^{ie\alpha/\hbar}\psi$ is a phase difference?

Background: magnetic monopole $\qquad \alpha = \oint_C \vec{A}\cdot d\vec{x} = \int_S d\vec{S}\cdot\vec{B}$



Solid angle Ω

$$\Rightarrow \alpha = \frac{\Omega g}{4\pi}$$

$$S' = S^2 - S$$

$$\Omega' = 4\pi - \Omega$$

$$\alpha' = -\frac{(4\pi - \Omega)g}{4\pi}$$

$e^{i\alpha e/\hbar} = e^{i\alpha' e/\hbar}$

$$\Rightarrow \frac{\alpha e}{\hbar} = \frac{\alpha' e}{\hbar} = (\alpha - g)\frac{e}{\hbar} \quad [2\pi]$$

$$\Rightarrow \frac{\alpha e}{\hbar} = (\alpha - g)\frac{e}{\hbar} + 2n\pi$$

$$\Rightarrow \boxed{ge = 2\pi n\hbar} \qquad n \in \mathbb{Z}$$

$$\Phi_0 = \frac{2\pi\hbar}{e}$$

"Quantum of flux"

F is the curvature of A → needs to be quantized?
F is a globally defined 2-form on M.
Moreover, dF is closed

$$\Rightarrow [F] \in H^2_{dR}(M)$$

with periods in $2\pi\mathbb{Z}$, i.e. $\forall\, S^2 \subset M$

$$\int_{S^2} F \in 2\pi\mathbb{Z}.$$

↳ Characteristic class.

In fact, give a $U(1)$-bundle E over M, there is an element $c_1 \in H^2(M,\mathbb{Z})$ (first Chern class) cf. $\mathrm{Vect}^1_{\mathbb{C}}(M) \xrightarrow{\sim} H^2(M,\mathbb{Z})$.

$\forall$ unitary connection d_A on E w/ curvature F,

$$[F] = 2\pi\, \widetilde{c}_1$$

where $\widetilde{c}_1$ is the image of c_1 under

$$H^2(M,\mathbb{Z}) \longrightarrow H^2_{dR}(M).$$

<u>Classifying space</u>: $H^2(M,\mathbb{Z}) \simeq [M, \mathbb{C}P^\infty]$

$$\swarrow$$

$K(U(1),1)$

(Eilenberg - McLane)

<u>Classification of $U(1)$ connections</u> (modulo gauge)

If $H^1(M;\mathbb{Z})$ is free abelian of rank b_1,

$$\Gamma : \mathcal{B} \longrightarrow \mathcal{E} \times \underbrace{S^1 \times \cdots \times S^1}$$

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$$\Gamma : \mathcal{B} \longrightarrow \mathcal{E} \times \underbrace{S^1 \times \dots \times S^1}_{b_1}$$

$$\Gamma([A]) = \left(\frac{1}{2\pi} F_A , e^{i\theta_1(A)}, \dots, e^{i\theta_{b_1}(A)}\right)$$

is a bijection.

<u>Moral</u>: A gauge field $A^{(1)}$ is a principle $U(1)$ connection, and in particular, it is <u>not</u> a differential 1-form.

However, on a sufficiently small neighborhood of every point of M, it can be represented as a 1-form:

$\Rightarrow$ $(\mathcal{U}_i)_{i\in I}$ open cover of M (paracompact)

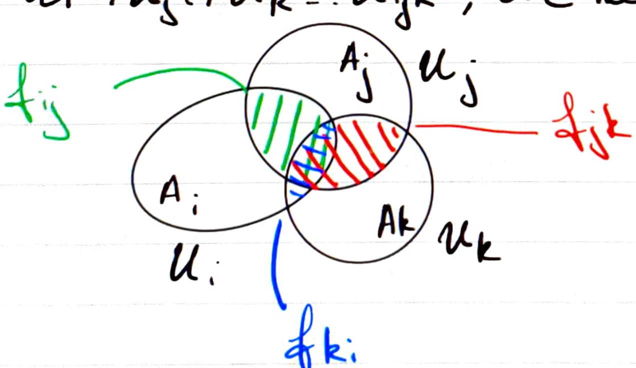
$A_i^{(1)} \in \Omega^1(\mathcal{U}_i, \mathbb{R})$.

$\forall i, j \in I$, there exists a function

$$f_{ij} : \mathcal{U}_i \cap \mathcal{U}_j \longrightarrow S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$$

s.t $A_i^{(1)} - A_j^{(1)} = df_{ij}$ on $\mathcal{U}_i \cap \mathcal{U}_j$.

On $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k =: \mathcal{U}_{ijk}$, one has a condition:



$$A_i - A_j = df_{ij} \; ; \; A_k - A_i = df_{ki} \; ; \; A_j - A_k = df_{jk}$$

$$\Rightarrow \quad f_{ij} + f_{jk} + f_{ki} = 2\pi m_{ijk}$$

& $m_{ijk} + m_{jkl} + m_{kli} + m_{lij} = 0$ on $\mathcal{U}_{ijkl}$.

<u>Good cover</u>: All $\mathcal{U}_{i_1} \cap \dots \cap \mathcal{U}_{i_k}$ are simply-connected. for every $k \geq 1$.

$dA^{(1)}$ is a globally defined closed 2-form.
Periods are constrained to by int. multiples of 2π.

<u>Genereralization</u>: higher form $U(1)$ gauge fields.

$$S = \int F^{(q+2)} \wedge * F^{(q+2)}$$

$F^{(q+2)} = d A^{(q+1)}$

<u>gauge</u>: $A^{(q+1)} \longrightarrow A^{(q+1)} + d\lambda^{(q)}$

LOCALLY

$\{U_\alpha\}_{\alpha \in A}$ open cover of M

$\forall \alpha \in A, \quad A^{(q+1)}_\alpha \in \Omega^{q+1}(U_\alpha, \mathbb{R})$

$\alpha, \beta \in A, \quad \lambda^{(q)}_{\alpha\beta} \in \Omega^q(U_\alpha \cap U_\beta, \mathbb{R})$

s.t. $A^{(q+1)}_\alpha = A^{(q+1)}_\beta + d\lambda^{(q)}_{\alpha\beta}$ on $U_\alpha \cap U_\beta$

On $U_\alpha \cap U_\beta \cap U_\gamma$:

$$\lambda^{(q)}_{\alpha\beta} + \lambda^{(q)}_{\beta\gamma} + \lambda^{(q)}_{\gamma\alpha} = d f^{(q-1)}_{\alpha\beta\gamma}$$

etc, etc ...

until $q = -1$.

<u>Example</u>: $(q=0)$ $\quad A_\alpha - A_\beta = d\lambda_{\alpha\beta}$ "circle valued" $\to$ real valued

$\lambda_{\alpha\beta} + \lambda_{\beta\gamma} + \lambda_{\gamma\delta} = 2\pi\, m_{ijk}$

$m_{ijk} \in \mathbb{Z}$ ...

Cocycle condition on quadruple overlaps?